

# Cubulations and hyperbolic manifolds

GGT Seminar January 2026

**Notation:** The \* above a topic means that it could be split into 2 talks, given jointly by a team of 2 people.

**Please see the end of this document for some more references on the topic.**

## 1 Introduction to CAT(0) cube complexes

Basic definition of cube complexes, criterion for when they are CAT(0), basic examples.

Hyperplanes, spaces with walls.

Right-angled Artin and Coxeter groups.

References:

- Bridson-Haefliger
- Sageev's notes at: [https://www.math.utah.edu/pcmi12/lecture\\_notes/sageev.pdf](https://www.math.utah.edu/pcmi12/lecture_notes/sageev.pdf)

## 2 Cubulations and partially ordered sets/spaces with walls

The idea is to go from an action on a space with walls to an action on a “dual” CAT(0) cube complex. I believe Sageev calls this space with walls a pocset in his lecture notes. We could read this stuff:

- either from Lecture 1 (Sec 3) and Lecture 2 (Sec 1,2) of Sageev's notes. Discuss examples from Lecture 2 Section 3.
- alternatively, we could follow the paper of Chatterji-Niblo, available at <https://arxiv.org/abs/math/0309036>.

### 3 Hyperbolic 3-manifolds, quasi-convex surface subgroups, subgroup separability \*

This could roughly be based on Lecture 2 of Sageev's notes cited above. The focus will be on Section 4.

Emphasize on the case of fundamental groups of compact hyperbolic 3-manifolds. Discuss quasi-convex surface subgroups in such hyperbolic 3-manifold groups. Explain with examples. State Kahn-Markovic's result that gives plenty of quasi-convex surface subgroups.

A good case to see could be hyperbolic 3 manifolds fibering over the circle. Give examples of non-quasi-convex surface subgroups.

Discuss the notions of subgroup separability and some related properties of hyperbolic 3 manifolds.

References: TBD. (See the end of this document).

### 4 Arithmetic hyperbolic manifolds\*

Introduce the notion of arithmetic lattices of  $SO(n, 1)$ . Discuss some explicit constructions. Discuss the ways to detect when an arithmetic lattice is uniform vs non-uniform.

Possible references: The book by Maclachlan-Reid titled 'The Arithmetic of Hyperbolic 3-manifolds'. A secondary reference could be some specific sections of Dave Witte Morris's book on Arithmetic Groups, where he discusses some examples of arithmetic lattices in  $SO(Q)$ .

### 5 Cubulating arithmetic hyperbolic manifolds\*

This talk will be based on: *Hyperplane sections in arithmetic hyperbolic manifolds* of Bergeron-Haglund-Wise. The url is: <https://londmathsoc.onlinelibrary.wiley.com/doi/abs/10.1112/jlms/jdq082>

The first goal is Theorem 1.10. In particular, the Step 1 in the proof Theorem 1.10 (see the remarks after Theorem 1.10). The second goal is to discuss how Theorem 1.10 gives a host of other conclusion about the homology of the underlying manifold and the separability properties.

### 6 Gromov Thurston manifolds

These are examples of negatively curved manifolds that cannot carry hyperbolic metrics. Classical reference is Gromov-Thurston's original paper titled *Pinching constants for hyperbolic manifolds*.

Available at <https://www.ihes.fr/~gromov/wp-content/uploads/2018/08/755.pdf>.

## 7 Cubulating Gromov-Thurston manifolds

Giralt's proof that Gromov-Thurston manifolds are cubulated.

Reference: <https://ems.press/content/serial-article-files/29822>.

## 8 Geometric condition for cubulation

Discuss Bergeron-Wise's geometric condition on cubulability via quasiconvex subgroups. The paper is *A boundary criterion for cubulation*, available at: <https://www.jstor.org/stable/23240564>.

## 9 Markovic's paper on a criterion for Canon conjecture

Goal is to read Markovic's paper connecting Canon conjecture to abundance of quasi-convex surface subgroups.

Reference: <https://link.springer.com/article/10.1007/s00039-013-0228-5>.

---

Below are some optional topics and directions we could take after we have learnt the basics as above.

## 10 Definitions of special cube complex and virtual specialness

Possibly, Lecture 4 of Sageev's notes.

## 11 Sketch of Agol-Wise's proof of Virtual fiber-ing theorem

A good starting point maybe Lorenzo Ruffoni's Master's thesis: [https://amslaurea.unibo.it/id/eprint/5637/1/ruffoni\\_lorenzo\\_tesi.pdf](https://amslaurea.unibo.it/id/eprint/5637/1/ruffoni_lorenzo_tesi.pdf).

## 12 Cubulation and obstruction to Property T

The introductory case could come from Bekka-de la Harpe's book of Property T. Discuss the Crofton formula of  $H^n$ .

Then discuss how the hyperplane structure for CAT(0) cube complexes aid in a coarse version of a Crofton formula. This second part could be based on <https://chatterj.perso.math.cnrs.fr/papers/CDHMedian.pdf>.

## 13 Boundaries of cube complexes

Hagen's paper comparing the simplicial boundary to the other ones, like Roller, etc.: [https://www.wescac.net/SimplicialBoundaryRevisedAGT\\_318fix.pdf](https://www.wescac.net/SimplicialBoundaryRevisedAGT_318fix.pdf)

---

Below are some more references on the topic.

## References (originating from the discussion on November 20)

'3-manifold groups' by Aschenbrenner-Friedl-Wilton  
<https://ems.press/content/book-files/23406> : Focusses on 3-manifod *groups* but has a nice outline with references of getting started on 3-manifolds and then move onto Agol-Wise Virtual Fiberings.

Also these notes by Hatcher seem to be a good place to read proofs of decomposition thms for 3-manifolds (including JSJ) and the Loop and Sphere theorem.  
<https://pi.math.cornell.edu/~hatcher/3M/3Mfds.pdf>