Talk 1: Introduction to Matrix Groups and Examples of Them

1 Matrix Groups

Definition 1.1. A subgroup G $\leqslant GL_n(\mathbb{K})$ which is also a closed Subspace is called a Matrix Group or a $\mathbb{K}$ -matrix Group [Ba, Prop. 1.30]

Not all Groups of Matrices are Matrix Groups!

Example 1.2. SL_n is a Matrix Group

Definition 1.3. for a Vector x $\in \mathbb{K}^n$ the lenght is defined as |x|=$\sqrt{(x_1)^2 + ... + (x_n)^2}$

Proposition 1.4. *The following Statements are equivalent: A is a linear Isometry, $Ax * Ay = x * y$, $A^T * A = I_n$ [Ba, Prop. 1.38]*

Lemma 1.5. $Isom_n(\mathbb{R}) = O(n) \ltimes Trans_n(\mathbb{R})$ $= \{AT : A \in O(n), T \in Trans_n\}$ *[Ba, Prop. 1.39]*

Lemma 1.6. *SU(2) is a double cover of So(3)*

Lemma 1.7. *The Group $Heis_3$ is not linear*

2 Overview of Groups

Overview 2.1. .
$GL_n(\mathbb{K}) = \{A \in M_n(\mathbb{K}) : det(A) \neq 0\}$
$SL_n(\mathbb{K}) = \{A \in M_n(\mathbb{K}) : det(A) = 1\}$
$UT_n(\mathbb{K}) = \{A \in GL_n(\mathbb{K})$:A is upper triangular$\}$
$SUT_n(\mathbb{K}) = \{A \in GL_n(\mathbb{K})$:A is unipotent$\}$
$O(n) = \{A \in GL_n(\mathbb{R}) : A^T A = I_n\}$
$SO(n) = \{A \in GL_n(\mathbb{R}) : A^T A = I_n, det(A) = 1\}$
U(n)=$\{A \in GL_n(\mathbb{C}) : A^* A = I_n\}$
SU(n)=$\{A \in GL_n(\mathbb{C}) : A^* A = I_n, det(A) = 1\}$
$Trans_n(\mathbb{K}) = \left\{ \begin{bmatrix} I & t \\ 0 & 1 \end{bmatrix} : t \in \mathbb{K}^n \right\}$
$\mathrm{Aff}_n(\mathbb{K}) = \left\{ \begin{bmatrix} A & t \\ 0 & 1 \end{bmatrix} : t \in \mathbb{K}^n, A \in GL_n(\mathbb{K}) \right\}$
$\mathrm{Isom}_n(\mathbb{K}) = \left\{ \text{f: } \mathbb{K}^n \to \mathbb{K}^n \text{ :f is an isometry} \right\}$

Example 2.2.
SO(2) = $\left\{ \begin{bmatrix} cos\theta & sin\theta \\ -sin\theta & cos\theta \end{bmatrix} : \theta \in [0,2\pi) \right\}$

SO(3) can be imagined as all the proper rotations of a Sphere

Exercise 2.3. Prove for that any Eigenvalue λ of a Matrix A $\in$ U(n) $|\lambda|$ =1

References

[Ba] Andrew Baker: *Matrix Groups.*

Topology (of Matrix Groups)
by Friedrich Homann talk #2

Topology What is topology?

A topology Let X be a set and let $\tau \subseteq P(x)$. Then τ is called a topology if:

i) Both the empty set and X are elements of τ.

ii) Any infinite union of elements of τ is an element of τ.

iii) Any intersection of finitely many elements of τ is an element of τ.

examples:

- chaotic / trivial / indiscrete topology
- discrete topology
- standart topology

Topological spaces Def.: the pair of a set and a topology on that set. A topological space is denoted (X, τ).

Open sets basic calculus notion: similar to open intervals.
Def.: U is an open set if and only if it is an element of the topology.
Therefore $GL_n(\mathbb{R}, \mathbb{C}) \subseteq M_n(\mathbb{R}, \mathbb{C})$ can be open subsets.

Closed sets Def.: A set P is closed if and only if the complement is open.

Continuity Def.: Let (M, τ_M) and (N, τ_N) be topological spaces. Then a map $f : M \to N$ is continuous if $\forall\ V \in \tau_N$: $\text{preim}_f(V) \in \tau_M$.
Theorem: the composition of continuous maps is continuous.

example:

- Let $S = \{1, 2, 3, 4\}$ be a set. $\tau = \{\emptyset, \{1\}, \{1, 2, 3, 4\}\}$ (*easy to check:*) τ is a topology on S.
- Furthermore: $\{1\}$ is a open set, $\{2, 3, 4\}$ is a closed set.
- Let $\tau' = \{\emptyset, S\}$ be a different topology on S. Let $f : (S, \tau) \to (S, \tau')$ be the identity map. Then f is continuous, but not its inverse, since the preimage of $\{1\}$ is not an open set with respect to τ'.

Compactness basic calculus notion: closed & bounded $\Leftrightarrow$ compact

Open cover Def.: If U is a family of open subsets u, then U is an open cover of a set E if $E \subseteq \bigcup\{u \mid u \in U\}$
example: $U = \{B_1(M, N) \mid M, N \in \mathbb{Z}\}$

Subcover Def.: V is a subcover opf U if V is a subset of U that also covers E.
Def.: E is compact if every open cover U has a finite subcover V.
Compactness is preserved by continuous functions.

Heine-Borel theorem: for any set S in $\mathbb{R}^n$, S is closed and compact $\Leftrightarrow$ S is compact i.e. every open cover has a finite subcover.
examples:

- $[a, b]$
- closed balls of finite radius
- $O(n)$ and $SO(n)$

non-examples:

- $\mathbb{R}$ (counterexample: $U = \{(-n, n) \mid n \in \mathbb{N}\}$, finitely many elements do not suffice)
- $(0, 1)$ (counterexample: $U = \{\frac{1}{n} \mid n \in \mathbb{N}\}$, again, finitely many elements do not suffice).

Conectedness Def.: not disconnected

Disconnectednes Def.: E is disconnected if there are nonempty, open and disjoint subsets of E such that the union of thos e subsets is E.
analogy/example: jigsaw puzzle
property: If $f : C \to f(C)$ is continuous and C is connected, then $f(C)$ is connected.
Therefore: a continuous function $f : (0, 1) \to (0, 0.5) \cup (1.5, 2)$ is impossible. *proof is left as an exercise/problem*
example:

- $\mathbb{R}$
- $GL_n(\mathbb{R})$ is disconnected because it has two disjoint components. The matrices with positive and the matrices with negative determinants.
- $GL_n(\mathbb{C})$ is connected.

Homeomorphisms *not* homomorphisms
a.k.a. the donut = coffee mug part of topology
property: homeomorphisms preserve the topological structure.
Def.: $f : M \to N$ is a homeomorphism if f is bijective and continuous "in both directions".
examples:

- $f : [0, 1] \to [0, 2]$ (f could be $x \mapsto 2x$)
- $(0, 1)$ and $\mathbb{R}$ are homeomorphic.

exercise:

a) *First find a homeomorphism between* $[0, 100]$ *and* $[0, 1]$.
b) *Then, find a homeomorphism between* $(0, 1)$ *and* $\mathbb{R}$.

Metric creates the notion of distance

has to meet certain properties:

i) $\mathrm{d}(x, y) \geq 0$ and $\mathrm{d}(x, y) = 0 \Leftrightarrow x = y$

ii) $\mathrm{d}(x, y) = \mathrm{d}(x, y)$

iii) $\mathrm{d}(x, y) + \mathrm{d}(y, z) \geq \mathrm{d}(x, z)$

A pair of a set S and a metric d denoted (S,d) is called a metric space.
Closely related: the notion of norm
examples:

- Euclidean metrix $\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$ in $\mathbb{R}^2$
- taxicab metric $|x_1 - x_2| + |y_1 - y_2|$ in $\mathbb{R}^2$
- L^P-metrics $(|x_1 - x_2|^P + |y_1 - y_2|^P)^{\frac{1}{P}}$ in $\mathbb{R}^2$

possible norm on $M_n(\mathbb{R})$: $\|A\| = \sup\{|Ax| : x \in \mathbb{R}^n, |x| = 1\}$
can be used to define a metric on $M_n(\mathbb{R})$ ($\mathrm{d}(A, B) = \|A - B\|$)

Subspace topology induction of topologies on subsets

Def.: Let (M, τ) be a topological space, $N \subset M$,
then $\tau|_N := \{u \cap N \mid u \in \tau\}$.

Proof that $\tau|_N$ is a topology is left as an exercise.

property: If N is an open set in M, then v is open in N if and only if it is open in M.
examples: We can equip $S_1 \subset \mathbb{R}^2$ with $\tau_{std.}|_{S_1}$