Maximal Tori

Definition 1. Let $r \geq 1$. The **standard torus of rank r** is defined as
$\mathbb{T}^r = \{\text{diag}(z_1, \ldots, z_r) : |z_1| = \ldots = |z_r| = 1\} \subseteq Gl_r(\mathbb{C})$.
A **torus of rank r** is any Lie group isomorphic to $\mathbb{T}^r$.

Definition 2. Let G be a Lie group. An element $g \in G$ is a (topological) generator of G, if $\overline{\langle g \rangle} = G$ with the cyclic subgroup $\langle g \rangle \subseteq G$.

Proposition 3. *Every torus has a generator.*

Definition 4. Let G be a Lie group and $T \subseteq G$ a closed subgroup which is also a torus. T is called a **maximal torus** in G, if the only torus $T' \subseteq G$ for which $T \subseteq T'$ is T itself.

Proposition 5. *(Standard maximal tori)*
Each of the following is a maximal torus in the stated group:
$\{R_{2n}(\theta_1, \ldots, \theta_n) : \forall k\ \theta_k \in [0, 2\pi)\} \subseteq SO(2n)$
$\{R_{2n+1}(\theta_1, \ldots, \theta_n) : \forall k\ \theta_k \in [0, 2\pi)\} \subseteq SO(2n+1)$
$\{\text{diag}(z_1, \ldots, z_n) : \forall k\ |z_k| = 1\} \subseteq U(n)$
$\{\text{diag}(z_1, \ldots, z_n) : \forall k\ |z_k| = 1, z_1 \cdots z_n = 1\} \subseteq SU(n)$

$$\textit{with } R(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

From now on let G be a compact, connected Lie group and $T \subseteq G$ a maximal torus.

Theorem 6. *For each $g \in G$ there exists an $x \in G$ such that $g \in xTx^{-1}$ (g is conjugate to an element of T). Equivalently, one can write $G = \bigcup_{x \in G} xTx^{-1}$.*

Theorem 7. *If $T, T' \subseteq G$ are maximal tori, there exists a $g \in G$ such that $T' = gTg^{-1}$.*

Theorem 8. *(Principal axis theorem)*
In each of the matrix groups $SO(n), U(n)$ and $SU(n)$ every element is conjugate to one element of the corresponding standard maximal torus.

Theorem 9. *(Principle axis theorem for Lie algebras)*
For each of the following Lie algebras $\mathfrak{g}$, every element $x \in \mathfrak{g}$ is conjugate in G to one of the stated form:
$\mathfrak{so}(2n):\ R'_{2n}(t_1, \ldots, t_n),\ \forall k\ t_k \in [0, 2\pi)$
$\mathfrak{so}(2n+1):\ R'_{2n+1}(t_1, \ldots, t_n),\ \forall k\ t_k \in [0, 2\pi)$
$\mathfrak{u}(n):\ \text{diag}(t_1 i, \ldots, t_n i),\ \forall t_k \in \mathbb{R}$
$\mathfrak{su}(n):\ \text{diag}(t_1 i, \ldots, t_n i),\ \forall t_k \in \mathbb{R}\ t_1 + \ldots + t_n = 1$

$$\textit{with } R'(t) = \begin{pmatrix} 0 & -t \\ t & 0 \end{pmatrix}$$

Definition 10. The **rank** of a compact Lie group is defined as the rank of a corresponding maximal torus.

Theorem 11. *Let $T \subseteq G$ be a maximal torus and $T \subseteq A \subseteq G$ where A is abelian. It follows that $A = T$ (i.e. every maximal torus is a maximal abelian subgroup).*

Definition 12. The **normaliser** $N_G(H)$ for a subgroup $H \subseteq G$ is the smallest subgroup of G in which H is normal: $N_G(H) = \{g \in G : \ gHg^{-1} = H\}$.
If $H = T$ is a maximal torus in G, the **Weyl group** $W_G(T)$ of T in G is defined as the quotient group $W_G(T) = N_G(T)/T$.

Theorem 13. *Let $T \subseteq G$ be a maximal torus.*

i) The Weyl group $W_G(T) = N_G(T)/T$ is finite.

ii) $W_G(T)$ acts on T by conjugation, i.e. $gT \cdot x = gxg^{-1}$. This action on T is faithful, meaning that the coset $gT \in N_G(T)/T$ acts trivially on T iff $g \in T$.

Exercise. Show that the standard maximal torus T for $U(2)$ given in Proposition 5 is indeed a maximal torus and calculate its Lie algebra as well as the normaliser of T in $U(2)$ and the corresponding Weyl group.

References

[1] Baker, Andrew *Matrix Groups: An Introduction to Lie Group Theory.*

[2] Tapp, Kristopher *Matrix Groups for Undergraduates.*