Seminar: Matrix Groups

Lie Bracket

May 4, 2023

Definition and Proposition (*Matrix Commutator*): Let $A \in M_n(\mathbb{K})$. The Matrix Commutator is defined as:

$$[\cdot, \cdot] : M_n(\mathbb{K}) \times M_n(\mathbb{K}) \to M_n(\mathbb{K}), [A, B] = AB - BA$$

The Matrix Commutator is bilinear, skew-symmetric and satisfies the Jacobi Identity.

Definition (*Lie Algebra*): A Lie Algebra $\mathfrak{g}$ is a $\mathbb{K}$-vector space together with the Lie Bracket: $[\cdot, \cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ satisfying Bilinearity, Skew-Symmetry and the Jacobi Identity.

Theorem 3.18[2]: If $G \in GL_n(\mathbb{K})$ is a matrix subgroup, then $\mathfrak{g} = T_I(G)$ is a $\mathbb{R}$ Lie Algebra in $M_n(\mathbb{K})$ with the Matrix Commutator as the Lie Bracket.

Definition and Proposition (*Lie Algebra Homomorphisms*): Let $\mathfrak{g}_1, \mathfrak{g}_2$ Lie Algebras of matrix groups G_1, G_2. A linear map $f : \mathfrak{g}_1 \to \mathfrak{g}_2$ is a Lie Algebra Homomorphism if:

$$f([A, B]) = [f(A), f(B)], \forall A, B \in \mathfrak{g}_1$$

If $f : G_1 \to G_2$ is a smooth group homomorphism then the derivative at identity $df_I : \mathfrak{g}_1 \to \mathfrak{g}_2$ is a Lie Algebra homomorphism.

We can see: Smoothly isomorphic matrix groups have isomorphic Lie Algebras, but the inverse is not true. An example for this is $SO(3)$ and $SU(2)$. Their Lie Algebras are isomorphic but $SU(2)$ is a double cover of $SO(3)$.

In addition to the last talk on the complexification of Lie Algebras note that $sl_2(\mathbb{R})$ is not isomorphic to $so(3)$ while their complexifications are isomorphic which we are now able to proof.

The *Lie correspondance theorem* is showing us the one-to-one connection between subgroups of $Gl_n(\mathbb{R})$ and subalgebras of $gl_n(\mathbb{R})$.

The adjoint representations of Lie Group and Algebra are a way to express elements of new or unknown Lie Groups with subgroups of $Gl_n(\mathbb{R})$ and $gl_n(\mathbb{R})$ which we already know.

Questions

1. We have seen that the matrix vector space over the quaternions i and j, $V = span\{(i),(j)\}_{\mathbb{R}}$, is not a Lie Algebra with the Matrix Commutator. Define a Lie Bracket such that V becomes a real Lie Algebra.

2. Are these Lie algebras isomorphic?

$$so(3) = span\left\{\begin{pmatrix}0 & 0 & 0\\ 0 & 0 & -1\\ 0 & 1 & 0\end{pmatrix}, \begin{pmatrix}0 & 0 & 1\\ 0 & 0 & 0\\ -1 & 0 & 0\end{pmatrix}, \begin{pmatrix}0 & -1 & 0\\ 1 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}\right\}$$

$$sp(1) = span\left\{\frac{1}{2}(i), \frac{1}{2}(j), \frac{1}{2}(k)\right\}$$

3. Can the $\mathbb{R}^3$ be seen as a Lie Algebra? What would be the Lie Bracket?

4. What exactly is the image of $ad: \quad X \in \mathfrak{g} \ \mapsto ad_X \in$ ___?

References

[1] Tapp, Kristopher *Matrix Groups for Untergraduates.* American Mathematical Society, [2016] Volume 79

[2] Baker, Andrew *Matrix Groups: An Introduction to Lie Group Theory.* Springer Undergraduate Mathematics Series