A GROUP THAT IS NOT LINEAR

1. Notation

Suppose G is a group.

(1) $Z(G)$ will be the center of G.
(2) If S is a subset of G, we will use $\langle S\rangle$ to denote the subgroup of G generated by (e.g. $\langle a\rangle$ is the cyclic subgroup of G generated by a).
(3) The commutator subgroup of G is the subgroup of G generated by the set $\{aba^{-1}b^{-1} : a, b \in G\}$.
(4) $\mathbb{T}$ will denote the circle group, i.e. $\{z \in \mathbb{C} : |z| = 1\}$.

2. Construction of the example: Heis_3

Consider the group

$$U_3 = \left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} : x, y, z \in \mathbb{R} \right\}.$$

Let $Z(U_3)$ be the center of the group U_3.

Lemma 2.1. $Z(U_3) = \left\{ \begin{pmatrix} 1 & 0 & z \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} : z \in \mathbb{R} \right\}.$

Proof. Exercise/ See Section 7.7 of Baker. □

Let Z_3 be the subgroup of $Z(U_3)$ with integer entries, that is,

$$Z_3 = \left\{ \begin{pmatrix} 1 & 0 & z \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} : z \in \mathbb{Z} \right\}.$$

Define

$$\mathrm{Heis}_3 := U_3/Z_3.$$

Remark 2.2. If this notation/definition G/H is unfamiliar to you, look up “Quotient Groups”.

Lemma 2.3. *$Z(\mathrm{Heis}_3)$, the center of Heis_3, is isomorphic to the circle $\mathbb{T}$. Moreover, every element in $Z(\mathrm{Heis}_3)$ in contained in the commutator subgroup of Heis_3.*

Proof. Exercise/ See Section 7.7 of Baker. □

3. The group Heis_3 isn't linear

We will spend the rest of this section proving the following result.

Theorem 3.1. *There does not exist any injective continuous group homomorphism $\phi : \mathrm{Heis}_3 \to \mathrm{GL}_n(\mathbb{R})$.*

We will prove this by contradiction. Suppose, on the contrary to theorem 3.1, that such a ϕ exists. Let

$$H := Z(\mathrm{Heis}_3).$$

By lemma 2.3, H is an infinite group. We will prove that $\phi(H) = \mathrm{Id}$ and thus arrive at a contradiction.

We first show that all elements of H are diagonalizable and have (possibly complex) eigenvalues of modulus 1.

Lemma 3.2. *There exists $g \in \mathrm{GL}_n(\mathbb{R})$ such that $\phi(H) \subset gO_ng^{-1}$ where O_n is the orthogonal subgroup.*

Proof. First explain why $\phi(H)$ must be compact (exercise).

Then use the following fact: **Any compact subgroup of $\mathrm{GL}_d(\mathbb{R})$ is contained in some conjugate of O_d. In fact the conjugates of O_d are the set of maximal compact subgroups of $\mathrm{GL}_d(\mathbb{R})$.** (Remark that we will discuss more about these things in the talk on ”Maximal Tori” later in the seminar). □

We will need to recall the following three results from linear algebra.

Proposition 3.3. *Any orthogonal matrix is diagonalizable over* $\mathbb{C}$*, that is, if* $A \in O_d$*, then there exist* $\mu_1, \ldots, \mu_n \in \mathbb{C}$ *and* $g_1 \in GL_n(\mathbb{C})$ *(all depending on* A*) such that* $A = g_1 \begin{pmatrix} \mu_1 & \ldots & 0 \\ \vdots & \vdots & \vdots \\ 0 & \ldots & \mu_n \end{pmatrix} g_1^{-1}$.

Proposition 3.4. *Suppose* $A, B \in \mathrm{GL}_d(\mathbb{C})$ *are commuting matrices and they are both diagonalizable. Then* A, B *are simultaneously diagonalizable, that is, we can choose a basis of* $\mathbb{C}^d$ *such that in that basis, both* A *and* B *becomes diagonal.*

Corollary 3.5. *Suppose* $\mathcal{I}$ *is some indexing set. If* $\{A_\alpha : \alpha \in \mathcal{I}\}$ *is a commuting family of diagonalizable matrices, then the family* $\{A_\alpha : \alpha \in \mathcal{I}\}$ *is simultaneously diagonalizable.*

Recall the notation that $\mathbb{T}$ is the circle group.

Lemma 3.6. *Let* g *be as in lemma 3.2. There exist group homomorphisms* $\mu_i : H \to \mathbb{T}$ *for* $1 \leq i \leq n$ *such that* $\phi(h) = g \begin{pmatrix} \mu_1(h) & \ldots & 0 \\ \vdots & \vdots & \vdots \\ 0 & \ldots & \mu_n(h) \end{pmatrix} g^{-1}$.

Proof. Exercise. Use the above linear algebra facts and lemma 3.2. □

By using the above result, we now show that the any element in the commutator subgroup maps to Id.

Lemma 3.7. *For any* $a, b \in \mathrm{Heis}_3$*,* $\phi(aba^{-1}b^{-1}) = \mathrm{Id}$*. In particular,* $\phi(y) = \mathrm{Id}$ *for any* y *in the commutator subgroup of* Heis_3.

Proof. For any $a, b \in \mathrm{Heis}_3$,

$$\phi(aba^{-1}b^{-1}) = g \begin{pmatrix} \mu_1(a)\mu_1(b)\mu_1(a^{-1})\mu_1(b^{-1}) & \ldots & 0 \\ \vdots & \vdots & \vdots \\ 0 & \ldots & \mu_n(a)\mu_n(b)\mu_n(a^{-1})\mu_n(b^{-1}) \end{pmatrix} g^{-1}.$$

Since $\mathbb{R}$ is commutative, $\mu_1(a)\mu_1(b)\mu_1(a^{-1})\mu_1(b^{-1}) = \mu_1(a)\mu_1(a^{-1})\mu_1(b)\mu_1(b^{-1}) = 1$. Similar reasoning applies to all other diagonal entries. Hence $\phi(aba^{-1}b^{-1}) = g\,\mathrm{Id}\,g^{-1} = \mathrm{Id}$.

The in particular part follows by definition of the commutator subgroup. □

By lemma 2.3, H is a subroup of the commutator subgroup of Heis_3. Thus $\phi(H) = \mathrm{Id}$.