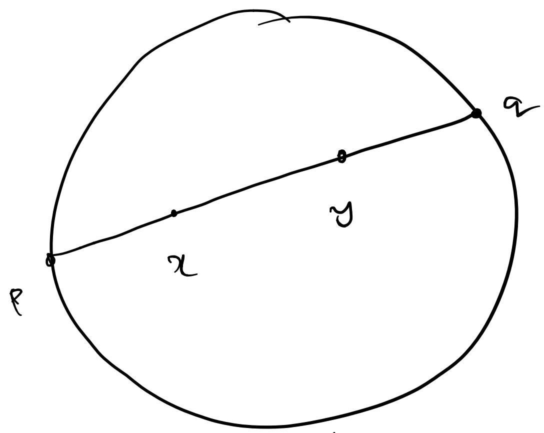
Hilbert Geometry

I Introduction

Def^n (Hilbert metric) $C \subseteq \mathbb{R}^n$ bdd convex.



$d_C(x,y)$

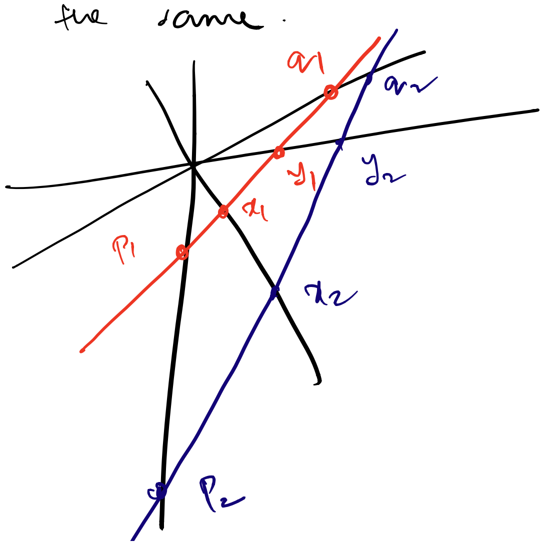
$= \frac{1}{2} \log \left(\frac{|p-y|\,|q-x|}{|p-x|\,|q-y|} \right)$

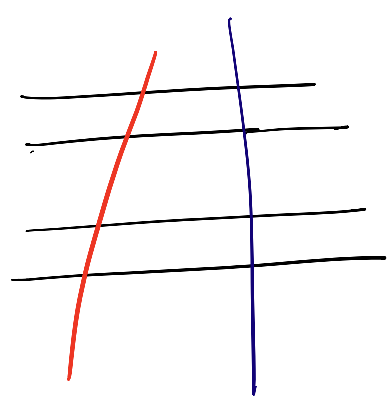
$= CR(p, x, y, q)$

d_C is indeed a metric, geodesics are st. lines (in a metric sense).

Recall a property of CRs:

For any 4 lines in a plane, meeting at a pt (maybe ∞), CR determined by any 2 lines not passing through intersection of the 4 is the same.





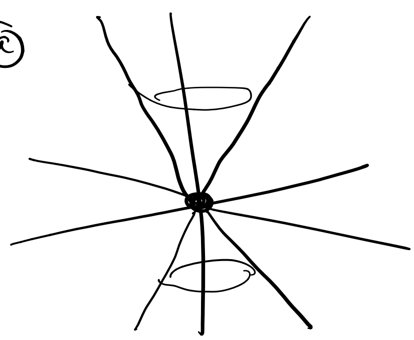
$$CR(p_1, x_1, y_1, q_1) = CR(p_2, x_2, y_2, q_2)$$

Def^n (properly convex) $\Omega \subseteq \mathbb{RP}^n \equiv \mathbb{P}(\mathbb{R}^{n+1})$ is convex and $\overline{\Omega}$ is properly convex if $\overline{\Omega}$ misses a projective hyperplane.

projectivization of a hyperplane in $\mathbb{R}^{n+1}$.

Example:

ⓐ

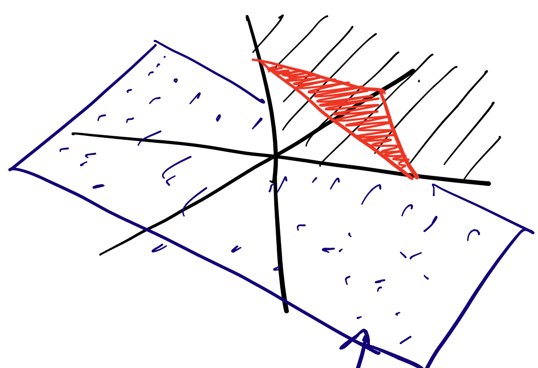


Take a cone C in $\mathbb{R}^3 \setminus 0$. $\overline{C}$ misses the hyperplane $\mathbb{R}^2 \times 0 \setminus 0$. Then, projectiviation gives $\Omega = \mathbb{P}(C)$, a properly convex set.

ⓑ $C = \mathbb{R}^+ e_1 \oplus \mathbb{R}^+ e_2 \oplus \mathbb{R}^+ e_3 \setminus 0.$

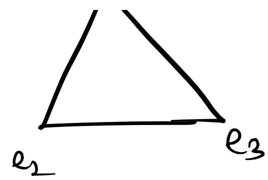
So, C properly convex. Projective C, by looking at the plane $x + y + z = 1$.

we get a triangle,



call it T ... e_1

call it ≡



(will be important later)

ⓒ Non - example

$C =$ upper half space in $\mathbb{R}^3$

↓ intersects every hyperplane.

So, $\Omega = \mathbb{P}(C)$ is NOT properly convex.

Why properly convex sets?

Ans → admits Hilbert metric. Let $\Omega = \mathbb{P}(C)$, C misses a hyperplane. Say it misses $x_{n+1} = 0$.

Then, $\Omega = \mathbb{P}(C)$, where

$$C \subseteq \{(x_1, \dots, x_{n+1}) \mid x_{n+1} \neq 0\}$$

↑ is a projective chart.

and the map

$$\underset{\mathbb{R}^{n+1}}{(x_1, \dots, x_{n+1})} \longmapsto \underset{\mathbb{R}^n}{\left(\frac{x_1}{x_{n+1}}, \dots, \frac{x_n}{x_{n+1}}\right)}$$

induces a homeomorphism from Ω to a "bounded convex subset of $\mathbb{R}^n$."

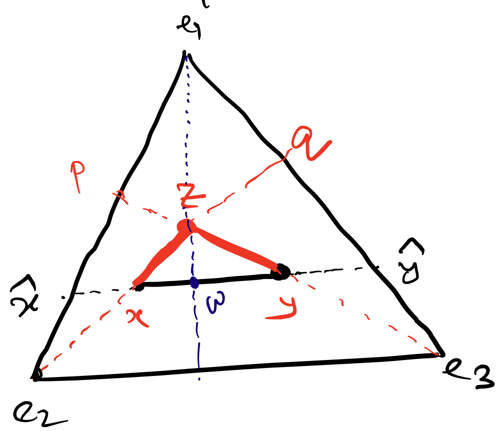
Hence, think of $\underset{(\subseteq \mathbb{RP}^n)}{\Omega} \Leftrightarrow$ bdd convex subset of $\mathbb{R}^n$

So, we can define a Hilbert metric on Ω, d_Ω, using these "affine charts".

Since $\overline{C}$ stays away from $x_{n+1} = 0$, its image $(\overline{\Omega} = \mathbb{P}(\overline{C}))$ in affine chart is bounded.

Interesting properties of geodesics:

① Non uniqueness.



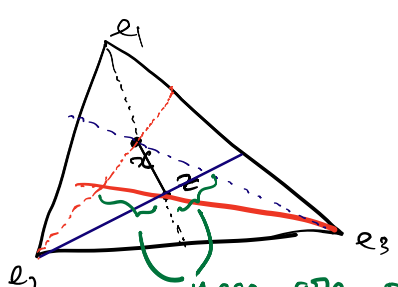
$$d_\Omega(x,w) = CR(\hat{x}, x, w, \hat{y})$$
$$= CR(e_2, x, z, q)$$
$$= d_\Omega(x,z)$$

$$d_\Omega(w,y) = d_\Omega(z,y)$$

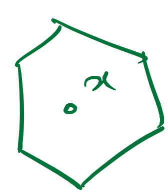
$$d_\Omega(x,y) = d_\Omega(x,z) + d_\Omega(z,y).$$

② shape of unit ball. (depends on shape of $\partial\Omega$)

For a T, unit ball is hexagonal.



these are pts at unit dist. from x (if $d_T(x,z) = 1$).



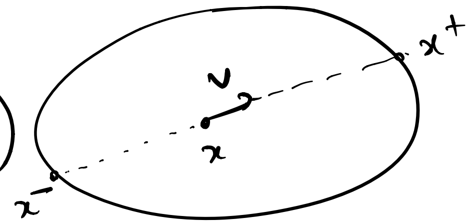
unit ball around x in T.

③ d_Ω is a Finsler metric (not Riemannian in general)

Let $v \in T_x\Omega$.

Fix a Euclidean norm $\|\cdot\|$.

Then, $F_x(v) = \frac{\|v\|}{2}\left(\frac{1}{\|x^+ - x\|} + \frac{1}{\|x^- - x\|}\right)$



$\{F_x\}_{x\in\Omega}$ induces d_Ω on Ω.

$\boxed{Q}$ When are these domains Gromov hyperbolic?
NICE ANS w/ DIVISIBILITY ASSUMPTION.

Divisible convex sets:

Let $\Omega \subseteq \mathbb{R}\mathbb{P}^n$ be a properly convex set.

$PSL(n+1, \mathbb{R})$ acts on $\mathbb{R}\mathbb{P}^n$.

$Aut(\Omega) = \{ g \in PSL(n+1, \mathbb{R}) \mid g(\Omega) = \Omega \}$.

and $Aut(\Omega) \subseteq Isom(\Omega)$.

<u>Def^n</u> (divisible) Ω is divisible if $\exists\ \Gamma \leq Aut(\Omega)$, discrete such that $\Gamma \backslash \Omega$ is compact.

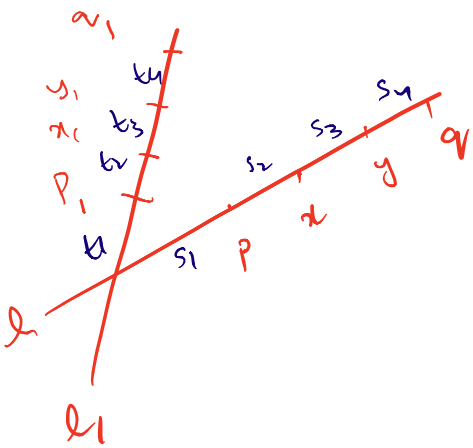
<u>NOTE:</u> Proof of $Aut(\Omega) \subseteq Isom(\Omega)$.

Since $g(\Omega) = \Omega$, g is actually an affine map between images of Ω in affine chart $\mathbb{A}^n$. So, g is a translation & a linear map.

Also, affine map takes lines to lines.

But translation doesn't alter cross-ratios.

Now, need to check if linear map changes cross-ratios.



Let $l_1 = A\,l$,

$A \to$ linear.

So, we have reduced our problem from dim n to dim 2 (plane containing l_1 and l)

In this plane choose basis $e_1 = l$, $e_2 = l_1$.

Then, $A \equiv \begin{pmatrix} 0 & \mu \\ \lambda & \sigma \end{pmatrix}$.

Easy to observe that

$t_1 = \lambda s_1$, $\quad t_1 + t_2 = \lambda(s_1 + t_2)$, etc.

Hence, $CR(p_1, x_1, y_1, q_1) = CR(p, x, y, q)$.

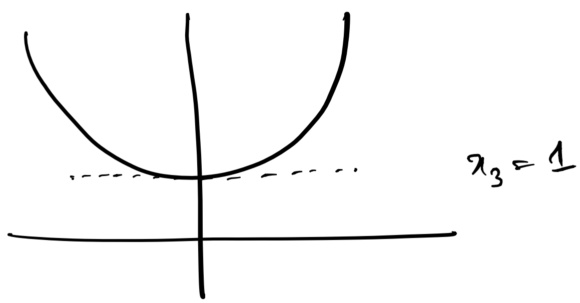
So, we have the required proof.

__Related Th__ (T. Speer): If Ω is bdd convex set in $\mathbb{R}^n$, then either $PGL(\Omega) = Isom(\Omega)$ or $Isom(\Omega)/PGL(\Omega) \cong \mathbb{Z}_2$.

II Examples :

A) Hyperbolic n-spaces.

$\mathbb{H}^2$ Beltrami - Klein model.



$x_3^2 - x_1^2 - x_2^2 = 1$.

project to plane $x_3 = 1$.

$$(x_1, x_2, x_3) \longmapsto \left(\frac{x_1}{x_3}, \frac{x_2}{x_3}, 1\right)$$

So, $\left(\frac{x_1}{x_3}\right)^2 + \left(\frac{x_2}{x_3}\right)^2 = 1 - \frac{1}{x_3^2}$.

$\Rightarrow$ Image of hyperboloid is a projective disk.

geodesics $\qquad$ (st. line)

$$\left[\ (\sinh t, 0, \cosh t) \longmapsto \left(\frac{\sinh t}{\cosh t}, 0, 1\right) = \left(\frac{e^t - e^{-t}}{e^t + e^{-t}}, 0, 1\right)\ \right]$$

$\mathbb{D} \subseteq \mathbb{RP}^2$ is the projective disk.

Induced Riem. metric on $\mathbb{D}$ coincides w/ Hilbert metric $d_{\mathbb{D}}$.

So, $\mathrm{Aut}(\Omega) \cong PSL(2, \mathbb{R})$, $\Omega = \mathbb{D}$.

Γ divides Ω iff Γ co-cpt lattice in $PSL(2, \mathbb{R})$.

NOTE: $\partial\Omega = \partial\mathbb{D}$ has no straight lines (in affine chart)

(B) Symmetric spaces (of non-cpt type).

$$SL(3,\mathbb{R})/SO(3) \underset{\text{homeo.}}{\cong} Pos_3^{tr} = \left\{ \begin{array}{l} 3\times 3 \text{ pos. def. symm} \\ \text{matrices with} \\ tr = 1 \end{array} \right\}$$

$SL(3,\mathbb{R})$ acts on Pos_3^{tr} by.

$$g \cdot A = \frac{gAg^t}{tr(gAg^t)}.$$

Obs that the action is transitive.

Fix basept $x_0 = \begin{pmatrix} 1/3 & & \\ & 1/3 & \\ & & 1/3 \end{pmatrix}$.

For any $A \in Pos_3^{tr}$, $\exists\, g$ st. $gAg^t = \begin{pmatrix} \lambda_1 & & \\ & \lambda_2 & \\ & & \lambda_3 \end{pmatrix}$,

$\lambda_i > 0$. Then, $h = \begin{pmatrix} \frac{1}{\sqrt{\lambda_1}} & & \\ & \frac{1}{\sqrt{\lambda_2}} & \\ & & \frac{1}{\sqrt{\lambda_3}} \end{pmatrix}$

$$\Rightarrow \quad hgAg^th^t = x_0$$

$$Stab_{SL(3,\mathbb{R})}(x_0) = \left\{ g \in SL(3,\mathbb{R}) \;\middle|\; gg^t = tr(gg^t)\, I \right\}$$

* Taking det, $tr(gg^t) = 1$

$$= SO(3).$$

Hence, $SL(3,\mathbb{R})/SO(3) \underset{\text{homeo}}{\cong} Pos_3^{tr}$.

Sym := 3x3 symmetric matrices of tr = 1

= 5 dimensional affine space ($\mathbb{P}(\mathbb{R}^6)$).

Pos_3^{tr} is open & convex in Sym.

since $Pos_3^{tr} = \left\{ a_{11} > 0, \begin{vmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{vmatrix} > 0, \det(A) > 0 \right\}$

Pos_3^{tr} is also bounded in Sym (affine chart):

$$A = \begin{pmatrix} x & b & c \\ b & y & d \\ c & d & z \end{pmatrix} \Rightarrow \begin{matrix} x > 0, \quad y > 0, \quad z > 0. \\ \begin{vmatrix} x & b \\ b & y \end{vmatrix}, \begin{vmatrix} y & d \\ d & z \end{vmatrix}, \begin{vmatrix} x & c \\ c & z \end{vmatrix} > 0 \end{matrix}$$

and $tr = 1 \Rightarrow x + y + z = 1$

$\Rightarrow$ $x, y, z \in [0, 1]$

$b, c, d \in [-1, 1]$.

Hence, Pos_3^{tr} can be equipped with Hilbert metric.

☒ Pos_3^{tr} & $SL(3,\mathbb{R})/SO(3)$ have very different geometries.

$SL(3,\mathbb{R})/SO(3)$ is simply conn & non-pos. curved.

$\Rightarrow$ uniqueness of geodesics.

But Pos_3^{tr} has PETs $\Rightarrow$ non-unique geodesics.

PETs: Let $e_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

and all rk 1 matrices are $g e_1 g^t$.

Let e_2, e_3 be any 2 such distinct rk 1 matrices. Then, e_1, e_2, e_3 is a T w/ $\partial T \subseteq \partial \Omega$.

So, $d_\Omega\big|_T = d_T\big|_T \Rightarrow T$ is properly embedded.

[Observe that $\partial P_{os_3}^{tr}$ consists of semi-definite matrices. So, lot's of Δs in the boundary. But the entire boundary is not Ts $\rightarrow$ there are copies of $\mathbb{RP}^2$ (ie. hyperbolic slices).]

In general, this construction produces a Hilbert geometry on symmetric spaces.

? { By a thm of Benoist, Γ lattice in $SL(d+1, \mathbb{R})$ where $d = \dim(Pos_n^{tr})$. }

So, question: are there new examples of discrete groups Γ that show up, but are not lattices?

Ans: Yes, from exotic examples due to Benoist (low dims.) and Kapovich (all dim ≥ 4).

(c) $T = P(\mathbb{R}^+ e_1 \oplus \mathbb{R}^+ e_2 \oplus \mathbb{R}^+ e_3)$

$\mathbb{R}^2$ acts on T by diag subgrp of $SL(3, \mathbb{R})$.

Transitive action: For any $[a, b, c] \in T$, find k s.t. $(ka)(kb)(kc) = 1$.

Then, $\begin{pmatrix} ka & & \\ & kb & \\ & & kc \end{pmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$

Free action, so $\mathbb{R}^2 \underset{\text{homeo}}{\cong} T \Rightarrow \mathbb{R}^2 / \mathbb{Z}^2 \cong \mathbb{Z}^2 \backslash T$

where $\mathbb{Z}^2 = \left\{ \begin{pmatrix} 2^{m+n} & & \\ & 2^{-m} & \\ & & 2^{-n} \end{pmatrix} : m, n \in \mathbb{Z} \right\}$.

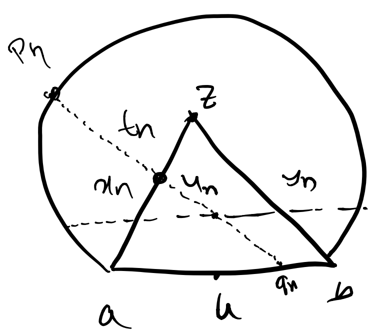
III Benoist's Results on Divisible Convex sets

strictly convex $\to$ no line segments in $\partial\Omega$

Th (Benoist): Ω divisible. Then Ω strictly convex $\iff$ Ω Gromov hyperbolic.

Proof: Ω Gromov hyp $\Rightarrow$ strictly convex (doesn't require divisible)

Let Maximal line segment $(a,b) \subseteq \partial\Omega$



Fix $z \in \Omega$, $u \in (a,b)$

$x_n \to a$, $y_n \to b$, $u_n \to u$.

$d(u_n, z) \to \infty$.

want to show, $d(u_n, [z,x_n] \cup [z,y_n]) \to \infty$.

Suppose, $d(u_n, [z,x_n]) \leq B$.

$\Rightarrow \exists\, t_n \in [z,x_n]$ s.t. $d(u_n,t_n) = d(u_n, [z,x_n]) \leq B$.

$t_n \to t$, & $t \notin \Omega$ (since $u_n \in \partial\Omega$)

$\Rightarrow t = a$.

Hence, $u_n t_n \to ua$ & since (a,b) maximal

$\Rightarrow$ $p_n \to a$

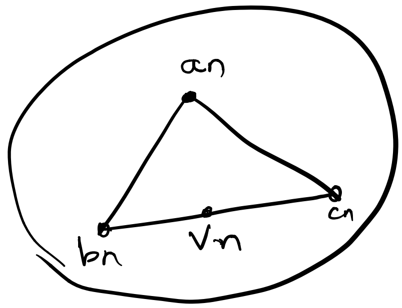
Hence, $d(u_n,t_n) = \log\left(\left(1+\frac{|u_n t_n|}{|t_n p_n|}\right)\left(1+\frac{|u_n t_n|}{|u_n q_n|}\right)\right)$

$|u_n t_n| \to |ua| \neq 0$, $|t_n p_n| \to 0$.

$\Rightarrow$ $d(u_n,t_n) \to \infty$, contradiction.

conversely, strict convexity $\Rightarrow$ Gromov hyp. (uses divisibility)

consider fat Δs



$d(v_n, [a_n, b_n] \cup [a_n, c_n]) \geq n.$

$v_n \to v$, $a_n, b_n, c_n \to a, b, c.$

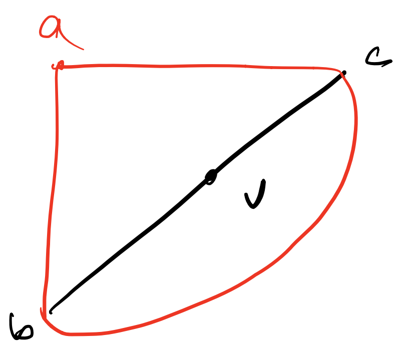
$\Rightarrow [a, b] \subseteq \partial\Omega$, $[a, c] \subseteq \partial\Omega$

$\because$ no line segment in $\partial\Omega$,

$a = b$, $a = c$.

$\Rightarrow b = c$, but $v \in (b, c)$,

contradiction.



<u>Cor:</u> $\Gamma \overset{QI}{\cong} \Omega \Rightarrow \Gamma$ Gromov hyperbolic.
if Ω strictly convex.

<u>Result:</u> strictly convex $\Omega \Rightarrow$ unique geodesics

<u>Th (Benoist)</u> : Ω = divisible, open properly convex domain. TFAE
Γ = torsion free dividing group.

(1) Ω strictly convex.
(2) Γ Gromov hyperbolic.
(3) $\partial\Omega$ is C^1
(4) The geodesic flow is Anosov.

<u>Pf:</u> (1) $\Leftrightarrow$ (2) above.

(1) $\Leftrightarrow$ (3) :

$\Gamma \curvearrowright \Omega \Rightarrow \Gamma^t$ acts on $\Omega^* := \{ f \in \mathbb{P}(V^*) \mid f(x) \neq 0 \ \forall\, x \in \overline{\Omega} \}$

Γ divides $\Omega \Leftrightarrow \Gamma^t$ divides Ω^*.

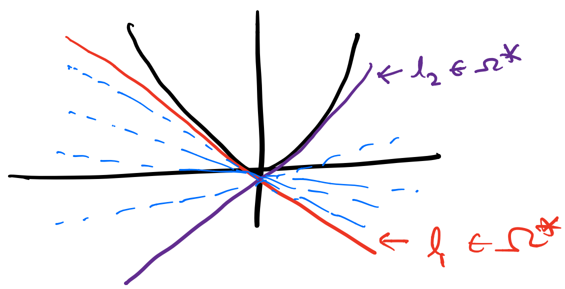
[$cd(\Gamma) = \dim \Omega = \dim \Omega^* = cd(\Gamma^t)$]

So, Ω strictly convex $\Leftrightarrow \Omega^*$ strictly convex

+

Ω^* strictly convex $\Leftrightarrow \partial\Omega$ is C^1

[If not C^1, say Ω not C^1 at origin



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these blue dotted lines produce a line in $\partial\Omega^*$ (it is clear they are in Ω^* as they intersect Ω only at origin)

(1) $\Leftrightarrow$ (4) is the hard part of Benoist I.

[<u>Def^n of geod flow</u> : $\omega = (x, \xi) \in T\Omega$.

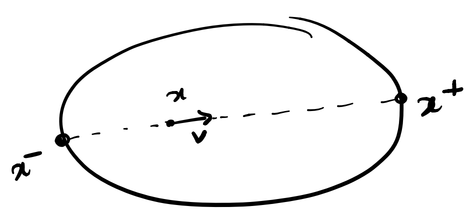
$\Phi_t(\omega) = (x_t, \xi_t)$ where

$$x_t = x + \frac{e^t - 1}{\delta_\omega^+ \cdot e^t + \delta_\omega^-} \xi$$

where $\delta_\omega^+, \delta_\omega^- \in \mathbb{R}$ s.t.

$$\xi = \delta_\omega^+ (x^+ - x)$$
$$\xi = \delta_\omega^- (x - x^-)$$

$\xi_t =$ derivative of x_t.



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Th (Benoist): For Ω divisible strictly convex, geod flow on $\Gamma\backslash\Omega$ is topologically mixing.

Both theorems are true for Riemannian negative curvature.

Cor: $\partial\Omega$ is more than C^1 $\to$ $\exists\, \alpha \in [1,2]$ and $\beta \in [2,\infty)$ such that $\partial\Omega$ is C^α regular and β convex.

$$\left\{ \begin{array}{l} \text{if } \partial\Omega \text{ is given by graph of } f(x) \text{ where } f(0)=0, \text{ then} \\ c_1 x^\beta \le f(x) \le C_2 x^\alpha . \end{array} \right.$$

Also, $\alpha_\Omega = \sup\{\alpha \in [1,2) \mid \partial\Omega \text{ is } C^\alpha\}$

$\beta_\Omega = \inf\{\beta \in [2,\infty) \mid \partial\Omega \text{ is } \beta\text{-convex}\}$.

$$\Rightarrow \frac{1}{\alpha_{\Omega^*}} + \frac{1}{\beta_\Omega} = 1 .$$

Results:

ⓐ Γ action on $\partial\Omega$ minimal

ⓑ If Ω is not an ellipsoid, Γ is Zariski dense in $SL(n+1,\mathbb{R})$. [If Ω ellipsoid, Γ lattice in $SO(n,1)$, hence not Z-dense in $SL(n+1,\mathbb{R})$]

Properties of dividing group Γ:

① All elements $g \in \Gamma - \{1\}$ are biproximal and g stabilizes a unique geodesic connecting x_g^+ and x_g^-, where x_g^+, x_g^- are pts in $\partial\Omega$ stabilized by g.

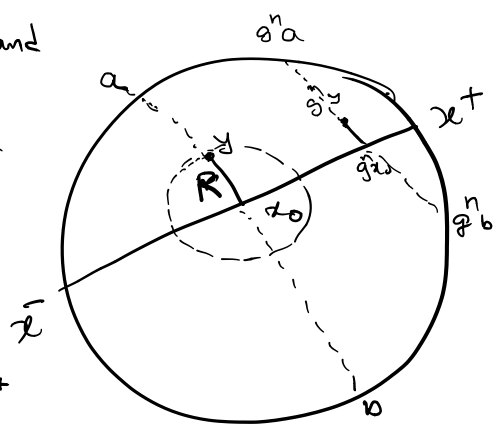
② Each free homotopy class $[g]$ contains a unique geod. representative. Length of this closed geod is,

$$\ell_{[g]} = \ell_1(g) - \ell_{n+1}(g)$$

③ If Ω is irreducible, not symmetric, then Γ is Zariski dense in $SL_{n+1}(\mathbb{R})$.

Proof:

① Lift $[g]$ to a geod $t \mapsto x_t$ in Ω. x^+, x^- are endpts of x. So, g acts as translation along x and fixes x^+, x^-.



Fix a R ball around $x_0 \in x$ & look at $g^n B(x_0, R)$.

obs: $g^n B(x_0, R) \to x^+$.

[$\because d(g^n y, g^n x_0) = \text{const}$, let $g^n y \to \bar{y}$

If $|\bar{y} - x^+| > 0$, $\lim g^n a \neq \bar{y}$, $\lim g^n b \neq x^+$

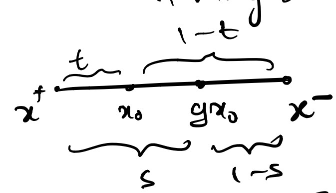
$\Rightarrow$ get a line in bdry through $\bar{y}$ and x^+

$\Rightarrow \bar{y} = x^+$.

This implies that $\lambda_1(g) > \lambda_2(g)$. Similarly, $\lambda_n(g) > \lambda_{n+1}(g)$.

② As $\Gamma \backslash \Omega$ compact, each homotopy class $[g]$ has a geod representative. Uniqueness of geodesics between pts in Ω (strict convexity implies this) implies uniqueness of rep.

For computing length, enough to look at a slice containing x^+ and x^-.



$$\ell_{[g]} = d_\Omega(x_0, gx_0) = \ln \frac{|gx_0 - x^+|\,|x_0 - x^-|}{|x_0 - x^+|\,|gx_0 - x^-|}$$

g restricted to this is $\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_{n+1} \end{pmatrix}$ $\Rightarrow$ $x_0 = (1-t)x^+ + t x^-$

$gx_0 = (1-s)x^+ + s x^-$

$= \lambda_1(1-t)x^+ + \lambda_{n+1} t x^-$

$\therefore \ell_{[g]} = \log \lambda_1 - \log \lambda_{n+1}$

IV Non-strictly convex case:

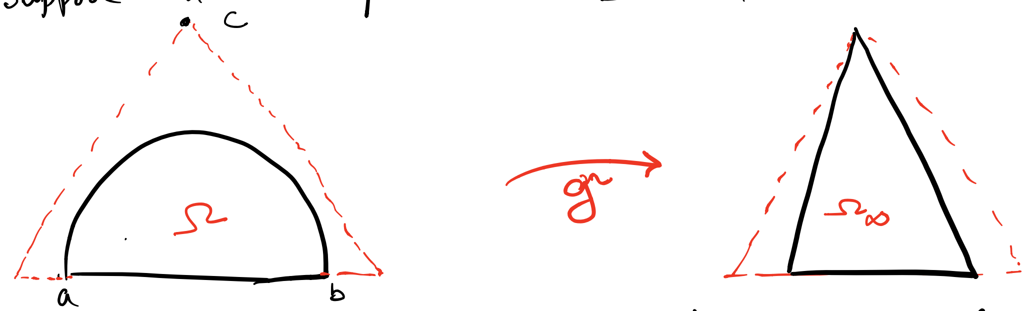
Now we want to prove results about divisible Ω where Ω is open, properly convex.

Warm up (dim 2)

Fact (Benzecri): $\mathcal{F}_m = \left\{ (\Omega, x) \;\middle|\; \begin{array}{l} \Omega \subseteq \mathbb{RP}^m \text{ open properly convex} \\ x \in \Omega \end{array} \right\}$

$PGL_{m+1}(\mathbb{R})$ acts on $\mathcal{F}_m$. Then, $\mathcal{F}_m / PGL_{m+1}$ is compact

Suppose non strictly convex Ω in dim 2.



Pick $g \in PGL_3(\mathbb{R})$ w/ eigenvectors a, b, c & eigenval (c) $>$ eigenval (b) $=$ eigenvalue (a).

$$[g\Omega] = [\Omega] \longrightarrow [\Omega_\infty]$$

$\therefore h\Omega = \Omega_\infty$ for some $h \in PGL_3(\mathbb{R})$.

ie, Ω is projectively a T.

In dim 2, dichotomy

$$\Omega$$

not strictly convex → $\Omega \cong T$

strictly convex ↓

$\Gamma\backslash\Omega$ is $g = 0, 1$ surface.
$g \neq 0$ as Ω not cpt.
So, $\Gamma\backslash T \underset{\text{homeo}}{\cong}$ torus

$\Gamma\backslash\Omega$ is a hyperbolic surface $\Rightarrow$ Ω hyperbolizable; $\Gamma\backslash\Omega$ higher genus surface.

$\downarrow$

But this example is "reducible".

So, in dim 2, irreducible properly convex divisible sets are strictly convex and hyperbolizable.

<u>One indication</u>: Properly Embedded Ts play the role of "totally-geodesic flats" and away from Ts, Ω looks negatively curved.

<u>Th (Benoist)</u> [dim 3].

Ω open properly convex irreducible, $\Omega \subseteq \mathbb{RP}^3$.
$\Gamma \subset SL(4, \mathbb{R})$ divides Ω. Let $\mathcal{T}$ = set of properly embedded triangles in Ω.

Γ_T = stabilizier of T in Γ.

Then

(1) $\forall\, T_1 \neq T_2$ in $\mathcal{T}$, $\overline{T_1} \cap \overline{T_2} = \phi$
(2) Each $\mathbb{Z}^2$ subgp of Γ stabilizes some $T \in \mathcal{T}$.
(3) For all $T \in \mathcal{T}$, Γ_T contains $\mathbb{Z}^2$ as index 2 subgroup.
(4) Γ has finitely many orbits in $\mathcal{T}$.

(5) The triangles project to Klein bottle or tori in $\Gamma \backslash \Omega$ and there are finitely many of them. Cutting open $M = \Gamma \backslash \Omega$ along these tori / Klein bottles, we get hyperbolizable atoroidal pieces.

(6) Each line $\gamma \subseteq \partial \Omega$ is contained in ∂T for some $T \in \mathcal{T}$.

(7) If Ω is not strictly convex, the vertices of triangles for $T \in \mathcal{T}$ is dense in $\partial \Omega$.

Important corollaries: $\Gamma \curvearrowright \partial \Omega$ is minimal.

[Note that for Hilbert geometry model of symmetric spaces, $\Gamma \curvearrowright \partial \Omega$ is not minimal. But here, for irreducible, non-homogeneous examples, $\Gamma \curvearrowright \partial \Omega$ is minimal]

Similar results are not available for dim 4 or higher.

Coxeter group examples:

Dynamical Questions

- Riem neg. curv — geod flow + Liouville measure

$$\Big\downarrow \text{Anosov + loc. prod. structure.}$$

ergodic

- non pos curv — open question
- strictly convex case —

  Th (Benoist) : There is no geod flow inv. density in $S\Omega$ unless Ω = ellipsoid.

  { "Density" - means abs. cont. w.r.t. Leb meas, since Leb meas and Finsler vol. are in same meas class }

  But Th (Crampon, Benoist) : Meas. of max entropy exists + unique. Geod flow is ergodic w.r.t. this measure.

  (similar to negative curvature)

- non-strictly convex case (Harry's results) —

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Dynamical study of geodesic flow for Benoist 3 mflds

Constructs a Bowen - Margulis meas. on T^1M $\left(M = \Gamma\backslash\Omega\right)$ that is geod flow invariant.

This requires construction of Pat-Sul meas $\{\mu_x\}_{x\in\Omega}$

$$\mu_x = \lim_{s\to\delta_\Gamma^+} \mu_{x,s}$$

where $$\mu_{x,s} = \frac{1}{P(x,o,s)}\sum_{\gamma\in\Gamma} e^{-sd(x,\gamma x)}\,\delta_{\gamma x}$$

where $$P(x,y,s) = \sum_{\gamma\in\Gamma} e^{-s\,d(x,\gamma y)}\ ; \quad \delta_\Gamma = \inf\{s \mid P(x,y,s) < \infty\}$$

Then meas on $\partial\Omega\times\partial\Omega\setminus\Delta$

is $$d\bar{\mu}_x(v^-,v^+) = e^{2\delta\langle v^-,v^+\rangle_x}\, d\mu_x(v^-)\, d\mu_x(v^+)$$

$T^1\Omega \cong (\partial\Omega\times\partial\Omega\setminus\Delta)\times\mathbb{R}$

$\curvearrowleft$ Pat-Sul × Pat-Sul × Leb

{ meas on $T^1\Omega$

Pass to quotient

$\downarrow$

μ_{BM}

Th (Bray 2016/2017): on $\Gamma\backslash\Omega$ geod flow is ergodic w.r.t Bowen-Margulis meas.

Cor: This is a measure of maximal entropy.

[Maximal entropy = $h_{top} = h_{vol} > 0$
$= \delta_\Gamma$]

Q: Is this unique? Not known.

Drawback: works in dim 3 only.

convex co-compact real rk 1 :

$\Gamma \leq G$ discrete subgp , G real rk 1 simple Lie gp. , $X = G/K$.

TFAE

(i) $\Gamma \leq G$ convex co-cpt

(ii) $\Gamma \to X$ orbit map is QI embedding

(iii) Γ hyperbolic, $\exists$ inj, cont, Γ-equiv map $\xi : \partial\Gamma \to X(\infty)$.

- (Kleiner-Leeb) If G rk ≥ 2 + $\Gamma \leq G$ Z. dense in G $\Rightarrow$ Γ co-cpt lattice.

Projective Anosov reps : Γ word hyp,

$\rho : \Gamma \to PSL_{d+1}(\mathbb{R})$ is proj Anosov if

$\exists$ $\xi, \eta : \partial\Gamma \to \mathbb{P}(\mathbb{R}^{d+1})$, $\mathbb{P}((\mathbb{R}^{d+1})^*)$

s.t. (a) ξ, η Γ-equiv.

(b) For each $\gamma \in \Gamma$ infinite order w/ x_γ^+ attracting fixed pt of γ in $\partial\Gamma$, $\xi(x_\gamma^+)$, $\eta(x_\gamma^+)$ are attracting fixed pts of $\rho(\gamma)$ on $\mathbb{P}(\mathbb{R}^{d+1})$, $\mathbb{P}((\mathbb{R}^{d+1})^*)$.

(c) If $x \neq y \in \partial\Gamma$, $\xi(x) + \ker \eta(y) = \mathbb{R}^{d+1}$.

Th (DGK, Zimmer) : Let Γ nonelementary hyperbolic but not free or surface gp. If $\rho : \Gamma \to PSL_{d+1}(\mathbb{R})$ is proj - Anosov, then $\exists$ $\Omega \subseteq \mathbb{P}(\mathbb{R}^{d+1})$ properly convex ~~w/ $\rho(\Gamma)(\Omega) = \Omega$ such that $\rho(\Gamma)$~~

st $\rho(\Gamma) \curvearrowright \Omega$ is "regular" convex co-cpt.

Th (Zimmer) : If $\Lambda \leq \mathrm{Aut}(\Omega)$ discrete w/ Ω prop convex, $\Omega \subseteq \mathbb{P}(\mathbb{R}^{d+1})$ st. $\Lambda \curvearrowright \Omega$ "regular" convex co-compactly. Then $\rho : \Lambda \hookrightarrow PSL_{d+1}(\mathbb{R})$ is proj. Anosov.

Hilbert geometry and Anosov representations