Rigidity Theorems in Geometry

Abstract

The talk will be about some interesting rigidity theorems in geometry. I will begin with the Mostow Rigidity theorem which states that in dimension at least 3, closed hyperbolic manifolds that are homotopic are in fact isometric. The algebraic version states that (under certain conditions) isomorphism of lattices in semi-simple Lie groups extend to isomoprhism of the corresponding Lie groups. I will sketch a short proof of this theorem using an entropy rigidity theorem due to Besson-Courtois-Gallot. Some interesting properties of lattices in semi-simple Lie groups will follow. I also plan to outline some of the interesting ideas that go into Mostow's original proof. I will conclude with a brief discussion about Margulis's superrigidity and arithmeticity theorems.

Mostow Rigidity Theorem

Mostow rigidity theorem (Geometric version) :

Let M_1 and M_2 be two closed (compact, without boundary) hyperbolic manifolds of dimension $n \geq 3$ with isomorphic fundametal groups Γ_1 and Γ_2. Then, M_1 and M_2 are isometric.

Note 1 :

Here, $M_i = \mathbb{H}^n/\Gamma_i$. As universal covers of M_i are contractible, all homotopy groups are isomorphic and thus, M_i are homotopic. Thus, the above theorem is saying that in dim ≥ 3, homotopic closed hyperbolic manifolds are isometric. Also, the homotopy is homotopic to this isometry.

Note 2:

Although this is being stated for compact manifolds, the result can be extended to finite volume manifolds.

Note 3 :

Here, Γ_i are discrete co-compact subgroups of $G_i = \text{Isom}(\mathbb{H}^n) \simeq SO(1,n)$. According to Mostow rigidity, isomoprhism of Γ_i lifts to an automorphism of $SO(1,n)$.

Mostow rigidity theorem (Algebraic version) :

Let Γ_1 and Γ_2 be co-compact latties in $SO(1,n)$ that are isomorphic. Then, they are conjugate in $SO(1,n)$.

So, why the term rigidity?

Let M be a closed hyperbolic manifold of dimension ≥ 3, and let g_1, g_2 be two hyperbolic metrics on M. Mostow rigidity theorem implies (M, g_1) and (Mg_2) are isometric. So, the space of hyperbolic metrics on M has exactly one point. Hence the term rigidity.

In the algebraic setting this amounts to saying that, upto conjugacy, a discrete subgroup Γ has a unique co-compact embedding in $SO(1, n)$.

Why it fails for closed surfaces?

The answer comes from Teichmuller theory. For $g \geq 2$, $Teich(S_g)$ is the space of hyperbolic structures on S_g upto homotopy. It can be proved that this space is homemorphic to $\mathbb{R}^{6g-6}$. So, there are uncountably many different metrics on S_g although the fundamental group is always the same.

Geometric explanation (pair of pants construction of S_g) : Each hyperbolic metric on S_g is characteristized by $3g - 3$ length parameters and $3g - 3$ twist parameters – the parameters coming from the way you put togehter the pairs of pants to get S_g.

Algebraic explanation : $Teich(S_g)$ is homemorphic to the space of discrete, faithful representations of $\pi_1(S_g)$ on $PSL_2(\mathbb{R})$ upto conjugacy by $PGL_2(\mathbb{R})$. Any representation of $\pi_1(S_g)$ comes from choosing $2g$ SL_2 matrices as images of generators and incorporating 3 relations imposed by the product of commutators in $\pi_1(S_g)$. The quotient by PGL_2 introduces a drop in dimension by 3.

Entropy Rigidity Result (B-C-G) and its Application :

We will look first look at an entropy rigidity result due to Besson-Courtois-Gallot and then use it to obtain a short proof of Mostow rigidity theorem.

Volume entropy (definition) :

Let $\tilde{M}$ be the universal cover of (M, g). Then,

$$h_{vol}(M) = \lim_{R\to\infty} \frac{\log(\mathrm{Vol}(B(x, R)))}{R},$$

where $B(x, R)$ is the ball of radius R in the universal cover.

Proposition(Manning) :

$h_{vol}(M)$ exists and is non-negative for any compact manifold M.

BCG Thereom :

Let (M, g) and (N, g_0) be compact Riemannian manifolds of dimension $n \geq 3$ where g_0 is locally symmetric metric of negative curvature. Let $f : M \to N$ be a smooth map with $\deg(f) \neq 0$. Then,

$$h_{vol}(M)^n \mathrm{Vol}(M) \geq |\deg(f)| h_{vol}(N)^n \mathrm{Vol}(N).$$

The equality occurs if $(M, \lambda g)$ is locally symmetric and f is homotopic to a Riemannian covering map of (N, g_0) by $(M, \lambda g)$. (for some $\lambda > 0$)

Proof of Mostow Rigidity theorem from B-C-G Theorem

Let $f : (M_1, g_1) \to (M_2, g_2)$ is a homotopy equivalence. Then, $f^*\omega_2 = \omega_1$ and As f is a homotopy equivalence, $|\deg(f)| = 1$ and $Vol(M_1) = Vol(M_2)$. Then, BCG implies,

$$h^n_{vol}(M_1) = h^n_{vol}(M_2)$$

(as both metrics are hyperbolic, the inequality is true both ways). The equality case of BCG then implies, f is homotopic to a Riemannian covering(upto a scaling of g_1 by λ). Covering map that is homotopic to a homotopy equivalence is necessarily a bijection. Hence, the covering map is an isometry and $(M_1, \lambda g_1)$ is isometric to (M_2, g_2). But $Vol(M_1) = Vol(M_2)$ implies $\lambda = 1$ and hence we have the proof.

Generalization of Mostow Rigidity :

Instead of working with $SO(1, n)$, Mostow's rigidity theorem can be stated for general semi-simple Lie groups. The dimension restriction $n \geq 3$ now manifests in the form of a restriction on the groups that can appear as simple factors of the semi-simple group.

Isogeny (definition) :

Lie groups G and H are isogenous if their Lie algebras are isomorphic. The relationship of isogeny will be denoted by $G \simeq H$.

Generalizations of the rigidity theorem of lattices:

Theorem (Mostow, Prasad, Margulis)

Let

- G_1, G_2 be connected semi-simple Lie groups, with trivial center and no compact factors

- $\Gamma_1 \subset G_1, \Gamma_2 \subset G_2$ are isomorphic lattices
- no simple factor of G_i is isogenous with $PSL(2, \mathbb{R})$

Then, the lattice isomorphism extends to a continuous isomorphism between G_1 and G_2.

Note 1:

We have to exclude $PSL(2, \mathbb{R})$ - it is isogenous with $SO(1, 2)$. Hence, we are back in the case of 'compact surfaces' where Mostow rigidity fails.

Note 2:

It turns out that $G = SO(1, n)$, $n \geq 3$ is one of the most difficult cases. For other semi-simple Lie groups (precisely, those of rank ≥ 2), this theorem follows as a corollary of another rigidity theorem - Margulis superrigidity. But then, one needs to deal with rank 1 semi-simple groups separately.

Quasi-isometric rigdity :

Suppose G_i, Γ_i are as above. If in addition, Γ_i are irreducible, then a quasi-isometry between Γ_i lifts to an isomorphism between G_i.

Superrigidity

Motivation for superrigidity :

If $\Phi : \mathbb{Z}^n \to \mathbb{R}^m$ is a homomorphism, then it extends to a homomorphism from $\mathbb{R}^n \to \mathbb{R}^n$.

Superrigidity (Margulis) :

Let

- G_1, G_2 be connected semi-simple Lie groups, with trivial center and no compact factors
- Γ is an irreducible lattice in G_1
- $\Phi : \Gamma \to G_2$ is a homomorphism such that $\Phi(\Gamma)$ is Zariski dense in G_2
- G is NOT isogenous to a group of the form $SO(1, n) \times K$ or $SU(1, n) \times K$ for some compact group K.

Then, Φ extends to a continuous homomorphism $G_1 \to G_2$.

Superrigidity implies Mostow rigidity for irreducible lattices and when rank ≥ 2:

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Arithmeticity (Margulis)

Let G be a semi-simple group that is not isogenous with $SO(1,n) \times K$ or $SU(1,n) \times K$. Let Γ be an irreducible lattice in G. Then Γ is an arithmetic subgroup of G.

Arithmetic groups

Arithmetic subgroups are the ones obtained through "arithmetic" constructions : like $\mathbb{Z} \subset \mathbb{R}$, $SL(n,\mathbb{Z}) \subset SL(n,\mathbb{R})$.

Proposition: Let B be a symmetric bilinear form such that $B(\mathbb{Z}^n,\mathbb{Z}^n) \subset \mathbb{Z}$. Then, $SO(B)_{\mathbb{Z}}$ is a co-compact lattice in $SO(B)_{\mathbb{R}}$.

Philosophically, if $G \subset SL(n,\mathbb{R})$ is defined by polynomials over $\mathbb{Q}$, then $G_{\mathbb{Z}}$ is an arithmetic subgroup of G.

But let's consider a more interesting example where we will look at $G_{\mathbb{Z}[\sqrt{2}]}$. Define $B \equiv x^2 + y^2 - \sqrt{2}z^2$ and let $G = SO(B) \simeq SO(1,2)$. Look at $\Gamma = G_{\mathbb{Z}[\sqrt{2}]}$. It is not obvious that Γ is a lattice in G.

Consider the Galois embedding $\sigma(x + \sqrt{2}y) = x - \sqrt{2}y$. Observe that if g preserves B, $\sigma(g)$ preserves $\sigma(B) = x^2 + y^2 + \sqrt{2}z^2$, that is, $SO(\sigma(B)) \simeq SO(3)$. Also, $\Delta : \Gamma \to G \times SO(3)$ defined by (id, σ) has discrete image. As $SO(3)$ is compact, this implies Γ is discrete in the first factor G.

It can be checked that Γ is co-compact by using Mahler's crtierion or Godement's criterion. Godement would be as follows : For any $\gamma \in \Gamma$ look at $\sigma(\gamma)$. It cannot be unipotent as it lies in a compact group. So, eigenvlues $\neq 1$. Then, eigenvalues of γ are σ^{-1} of eigenvalues of $\sigma(\gamma)$, implying none of them are 1. So, its not unipotent. Hence Γ co-compact.

Thus, although it is not obvious that $G_{\mathbb{Z}[\sqrt{2}]}$ is a lattice in G, we can observe this looking at $G\times$ "Compact Group". This philosophy motivates the definiton of arithmetic subgroups.

Definition (Arithmetic groups) :

$\Gamma \subset G$ is arithmetic subgroup if :

- there exists $L \subset SL(n,\mathbb{R})$ defined over $\mathbb{Q}$
- there exists compact normal subgroups $K_L \subset L, K \subset G$
- group isomorphism $\phi : G/K \to L/K_L$
- $\overline{\phi(\Gamma)}$ is commensurable with $\overline{L_{\mathbb{Z}}}$ (the bars indicate images under quotienting by compact normal subgroups)

Outline of a more geometric proof of Mostow:

Uses three main ingredients :

- Role of quasi-isometries in $\mathbb{H}^n$
- Their extensions to $\partial\mathbb{H}^n$ that are quasi-conformal
- Ergodicity of the geodesic flow on $T^1(\mathbb{H}^n)$ (or, more generally, for rank 1 symmetric spaces)

Steps :

- From the homotopy equivalence $f : M_1 \to M_2$, construct $\tilde{f} : \mathbb{H}^n \to \mathbb{H}^n$ that is a quasi-isometry and is equivariant under action of Γ_1 and Γ_2, that is,

$$\tilde{f}(\gamma.x) = f_*(\gamma).\tilde{f}(x)$$

  for $\gamma \in \Gamma_1, x \in \mathbb{H}^n$. Use Milnor-Svarc lemma to construct this quasi-isometry. (**Let** Γ **act properly discontinuously, co-compactly and by isometries on a proper, geodesic metric space** X**. Then, (X,d) is quasi-isometric with** (Γ, d_{word}))

- Extend this quasi-isometry to boundary of $\mathbb{H}^n$ in the following way : For any geodesic ray σ of $\mathbb{H}^n$, $\tilde{f} \circ \sigma$ is a quasi-geodesic ray. Then, Morse-Mostow lemma implies that within a bounded distance from $\tilde{f} \circ \sigma$, lies a hyperbolic geodesic β. Then,

$$\tilde{f}([\sigma]) = [\beta].$$

- $\tilde{f}$ is equivariant under action of fundamental group on the boundary of $\mathbb{H}^n$. If $\tilde{f}([\sigma]) = [\beta]$, then

$$\tilde{f}(\gamma.[\sigma]) = \tilde{f}([\gamma.\sigma]) = [\delta] = [f_*(\gamma).\beta] = f_*(\gamma).[\beta].$$

- Denote by $\partial\tilde{f} : \partial\mathbb{H}^n \to \partial\mathbb{H}^n$ the restriction of $\tilde{f}$ to the boundary. It can be proved that $\partial\tilde{f}$ is a homemorphism and a quasi-conformal map $\to$ diferentiable almost everywhere. ($f : X \to Y$ is K-quasi conformal if for all $x \in X$, $f(B_r(x))$ is an ellipsoid whoe ratio of major and minor axis is bounded by K.)
- Use ergodicity of geodesic flow on $\mathbb{H}^n$ to prove that the quasi-conformal map $\partial\tilde{f}$ is actually conformal. Thus, it must be the restriction of an isometry of $\mathbb{H}^n$ to the boundary. This produces the necessary isometry between M_1 and M_2.