(I) Quasi-morphisms

Defⁿ: $\phi : \Gamma \to \mathbb{R}$ (or $\mathbb{Z}$ or vect-space V w/ Γ action) i.e, Γ module V

such that $\sup_{g_1, g_2 \in \Gamma} |\phi(g_1 g_2) - \phi(g_1) - \phi(g_2)| < \infty$

Group of quasimorphisms of Γ:

$$\widetilde{QM}(\Gamma) = \frac{\text{Quasi-morphism of } \Gamma}{(\text{Homomorphisms} \oplus \text{Bdd functions on } \Gamma)}$$

Properties:

(a) Every class $[\phi] \in \widetilde{QM}(\Gamma)$ has homogeneous representative.

[Homogeneous quasimorphism: A quasimorphism α s.t. $\alpha(g^n) = n\alpha(g)$ $\forall g \in \Gamma$.]

Let $\alpha(g) := \lim_{n \to \infty} \frac{\phi(g^n)}{n}$

[limit exists as $a_n = \phi(g^n)$ is subadditive with a bounded error, i.e., $a_{n+m} \leq a_n + a_m + E$, E indep of n, m]

if $|\phi(g_1 g_2) - \phi(g_1) - \phi(g_2)| \leq E$ $\forall g_1, g_2 \in \Gamma$,

then, $n\phi(g) - (n-1)E \leq \phi(g^n) \leq n\phi(g) + (n-1)E$.

$\Rightarrow \phi(g) - E \leq \alpha(g) \leq \phi(g) + E$

$\Rightarrow \phi - \alpha$ is bdd

$\Rightarrow [\phi] = [\alpha]$.

α is a quasimorphism as,

$$\phi((g_1 g_2)^n) \le n\,\phi(g_1 g_2) + (n-1)E$$
$$\le n\,(\phi(g_1) + \phi(g_2)) + nE$$
$$\le \phi(g_1^n) + \phi(g_2^n) + 3nE$$

$$\Rightarrow \quad \frac{\phi((g_1 g_2)^n) - \phi(g_1^n) - \phi(g_2^n)}{n} \le 3E$$

$$\Rightarrow \quad \alpha(g_1 g_2) - \alpha(g_1) - \alpha(g_2) \le 3E.$$

(Similarly, get a lower bound.) $\Rightarrow$ $|\alpha(g_1 g_2) - \alpha(g_1) - \alpha(g_2)|$ uniformly bounded.

(b) Homogeneous quasimorphisms are constant on conjugacy classes.

Observe that $\alpha(g^{-1}) = -\alpha(g)$, $\alpha(e) = 0$.

Thus, $|\alpha(ghg^{-1}) - \alpha(h)| \le E \quad \forall g, h \in \Gamma.$

But, $\alpha(ghg^{-1}) = \dfrac{\alpha(gh^n g^{-1})}{n}$

$\alpha(h) = \dfrac{1}{n}\alpha(h^n)$.

Then, $|\alpha(ghg^{-1}) - \alpha(h)| \le \dfrac{E}{n} \quad \forall\, n > 0$

$\Rightarrow \alpha(ghg^{-1}) = \alpha(h) \quad \forall g, h \in \Gamma.$

Examples of quasimorphism groups

① $\Gamma = \mathbb{Z}$.

Every $[\phi] \in \widetilde{QM}(\mathbb{Z})$ is represented by a homogeneous α. But $\alpha(n) = n\alpha(1) \Rightarrow \alpha$ is a homomorphism

Thus, $\widetilde{QM}(\mathbb{Z}) = \{0\}$

② $\widetilde{QM}(\mathbb{Z}^k) = 0$.

Can be done in 2 ways

— $\widetilde{QM}(\Gamma_1 \times \Gamma_2) \cong \widetilde{QM}(\Gamma_1) \times \widetilde{QM}(\Gamma_2)$

— $\alpha(m_1, \dots, m_k)$

$$= \sum_{i=1}^{k} \alpha(m_i e_i)$$

$$= \sum_{i=1}^{k} m_i \alpha(e_i) \qquad (\because \alpha \text{ is homogeneous})$$

is a homomorphism.

③ $\widetilde{QM}(\text{solvable}) = \{0\}$

④ $\widetilde{QM}(\text{amenable}) = \{0\}$.

⑤ $\widetilde{QM}(F_2) =$ infinite dimensional.

<u>Construction by Brooks</u>:

Let $w \in F_2 = \langle a, b \rangle$, Assume w cyclically reduced, i.e., ww reduced. Also, for simplicity, assume $w \neq a^k$ or b^l.

$f_w : F_2 \to \mathbb{R}$,

(count occurrences)

$f_w(g) = \#_w(g) - \#_{w^{-1}}(g)$ (without overlap)

<u>Obs:</u> $\mathrm{Hom}(F_2) = \mathbb{R} f_a + \mathbb{R} f_b$.

If $\varphi \in \mathrm{Hom}(F_2)$,

$$
\begin{aligned}
&\varphi(a^{i_1} b^{j_1} a^{-i_2} b^{-j_2}) \\
&= i_1 \varphi(a) + j_1 \varphi(b) - i_2 \varphi(a) - j_2 \varphi(b) \\
&= \varphi(a)(i_1 - i_2) + \varphi(b)(j_1 - j_2)
\end{aligned}
$$

$$f_a(a^{i_1} b^{j_1} a^{-i_2} b^{-j_2}) = i_1 - i_2$$

$$f_b(\quad - \quad) = j_1 - j_2$$

so, $\varphi = \varphi(a) \cdot f_a + \varphi(b) f_b$.

Then, we can check that for $w \neq a^k$ or b^l, f_w is NOT a homomorphism. (and in fact, a non-trivial quasimorphism)

If f_w is a trivial quasimorphism, then

$$f_w = \alpha f_a + \beta f_b + \delta, \quad \delta : F_2 \to \mathbb{R} \text{ a bdd function.}$$

Suppose $\alpha \neq 0$.

$$f_w(a^k) = \alpha k + \delta(a^k)$$

$$\Rightarrow \quad \delta(a^k) = -\alpha k \quad \Rightarrow \delta \text{ unbounded}$$

$\therefore \alpha = 0$.

similarly, $\beta = 0$.

$$\Rightarrow f_w = \delta \Rightarrow f_w(w^k) = \delta(w^k) = k, \text{ impossible}$$

Hence, f_w is a non-trivial quasimorphism.

we still need to check, $|f_\omega(g_1 g_2) - f_\omega(g_1) - f_\omega(g_2)|$ is unif. bdd.

Observe that if g_1 doesn't end in ω and g_2 doesn't begin with ω^{-1}, then

$$f_\omega(g_1 g_2) = f_\omega(g_1) + f_\omega(g_2) \pm 1$$

(since words at end of g_1 and start of g_2 might join to form ω or ω^{-1})

(since there is no reduction in $g_1 g_2$)

Otherwise, $g_1 = h_1 \omega\omega\omega$, $g_2 = \omega^{-1}\omega^{-1}\omega^{-1} h_2$.

$$f_\omega(g_1 g_2) = f_\omega(h_1) + f_\omega(h_2)$$

$$f_\omega(g_1) = f_\omega(h_1) + 3$$

$$f_\omega(g_2) = f_\omega(h_2) - 3$$

$$\Rightarrow f_\omega(g_1 g_2) = f_\omega(g_1) + f_\omega(g_2)$$

So, $|f_\omega(g_1 g_2) - f_\omega(g_1) - f_\omega(g_2)|$ bdd.

Note: This construction produces infinitely many quasi-morphisms, not necessarily independent.

Infact $\{f_\omega\}$ are not a basis (ref: Bucher's notes $\rightarrow$ linear dependence found by Grigorchuk)

(6) $\widetilde{QM}$ (hyperbolic groups) = infinite dimensional.

Similar construction.

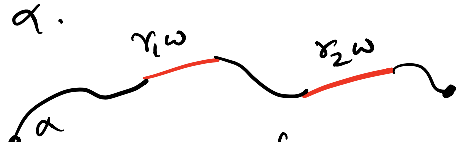
(Fujiwara)

(7) Suppose Γ acts on a Gromov hyperbolic space by isometries and the action is properly discontinuous. Assume that the limit set of the action Λ_Γ has at least 3 points (that is, Γ non-elementary). Then, there is an injective linear map

$$\omega : \ell^1(\Gamma) \longrightarrow H_b^2(\Gamma, \mathbb{R})$$

and $\widetilde{QM}(\Gamma)$ = infinite dimensional.

Construction: Let X = Gromov hyp. space, fix $x_0 \in X$. Fix a path ω in X. Define "copies of ω" to be the set of all Γ translates of ω. For any path α in X, let $|\alpha|_\omega$ = no. of copies of ω in α.



Here, $|\alpha|_\omega = 2$.

Then, for a fixed path ω in X, define $f_\omega : \Gamma \to \mathbb{R}$

$$f_\omega(g) := c_\omega(g) - c_{\omega^{-1}}(g)$$ where

$$c_\omega(g) = \inf_{\alpha \in P_{\omega,g}} \left(|\alpha| - |\alpha|_\omega \right)$$

where the infimum is taken over $\alpha \in P_{\omega,g}$ = set of all paths between x_0 and $g \cdot x_0$ in X.

and $|\alpha|_\omega$ = no. of "copies of ω" in α.
(as above)

Fujiwara identifies elements $\{g_i\}_1^\infty \in \Gamma$ s.t. $|[x_0, g_i \cdot x_0]| \geq C$ for some unif. constant C, and constructs f_{ω_i} where $\omega_i := [x_0, g_i \cdot x_0]$.

For an appropriate choice of g_i, this gives a basis of $\widetilde{QM}(\Gamma)$.

(II) Connections with group cohomology

Let $C^n(\Gamma, V) = \{f : \Gamma^{n+1} \to V\}$

$V = \mathbb{R}, \mathbb{Z}$ or a Γ-module V.

$$\delta^n : C^n \to C^{n+1}$$

$$\delta^n f(g_0, \ldots, g_{n+1}) = \sum_{i=0}^{n+1} (-1)^i f(g_0, \ldots, \hat{g_i}, \ldots, g_{n+1})$$

Consider the sub-complex of Γ-invariant objects.

$$D^n(\Gamma, V) = C^n(\Gamma, V)^\Gamma$$

$$\gamma \cdot f(g_0, \ldots, g_n) = f(\gamma^{-1} g_0, \ldots, \gamma^{-1} g_n).$$

(Ordinary) group cohomology of $\Gamma = H^*(\Gamma, V)$)

= cohomology of $(D^n(\Gamma, V), \delta^n)$

Suppose V has a norm $\|\cdot\|$. Then we can consider bounded cohomology.

$$C^n_b(\Gamma, V) = \{ f \in C^n(\Gamma, V) \mid \|f\|_\infty < \infty \}$$

where $\|f\|_\infty = \sup_{g_0, \dots, g_n \in \Gamma} \|f(g_0, \dots, g_n)\|$

Bounded cohomology of $\Gamma = H^*_b(\Gamma, V)$

$=$ cohomology of $\left(D^n_b(\Gamma, V), \delta^n \right)$

<u>Comparison map</u>:
$$H^n_b(\Gamma, V) \xrightarrow{c^n} H^n(\Gamma, V)$$
$$[f] \longrightarrow [f].$$

<u>Prop</u>: $\widetilde{QM}(\Gamma) = \operatorname{Ker}\left(H^2_b(\Gamma, \mathbb{R}) \xrightarrow{c^2} H^2(\Gamma, \mathbb{R}) \right)$

<u>Proof</u>: · First, talk about bar resolution of the complex $D^n(\Gamma, V)$. Elements of $H^2_b(\Gamma, \mathbb{R})$ are represented by functions $\Gamma^3 \to \mathbb{R}$. Bar resolution is a systematic way of thinking of these as functions $\Gamma^2 \to \mathbb{R}$.

$f \in D^n_1(\Gamma, V) \Rightarrow f : \Gamma^{n+1} \to V$

$f(g_0, \dots, g_n) = f(1, g_0^{-1} g_1, \dots, g_0^{-1} g_n)$

$= \bar{f}(\dots) \in C^{n-1}(\Gamma, \dots)$

$\longrightarrow f'(g_0^{-1}g_1, \ldots, g_0^{-1}g_n) \in C^{n}(\Gamma, V)$

We say, $\overline{D}^n(\Gamma, V) = C^{n-1}(\Gamma, V)$.

δ^n can be suitably modified to get a differential $\overline{\delta}^n$ in $\overline{D}^n$.

Thus, $H^n_b(\Gamma, \mathbb{R})$ = cohomology of $(\overline{D}^n_b(\Gamma, V), \overline{\delta}^n)$.
$H^n(\Gamma, \mathbb{R})$ = " " $(\overline{D}^n(\Gamma, V), \overline{\delta}^n)$

In particular,

$$H^2(\Gamma, \mathbb{R}) = \frac{\{f : \Gamma^2 \to \mathbb{R} \mid \overline{\delta}^2 f = 0\}}{\{g : \Gamma^2 \to \mathbb{R} \mid \underset{\text{w/ } h}{g} = \overline{\delta}_1 h\}}$$

and $H^2_b(\Gamma, \mathbb{R})$ = same thing but with bdd functions.

Now, the proof:

$$\begin{array}{ccc} \widetilde{QM}(\Gamma) & \xrightarrow{F} & H^2_b(\Gamma) \\ {[f]} & \longmapsto & [\overline{\delta}^1 f] \end{array}$$

$\overline{\delta}^2 \overline{\delta}^1 f = 0 \Rightarrow \overline{\delta}^1 f$ indeed represents a cohomology class.

Also, if f is a homomorphism,

$$\overline{\delta}^1 f(g_0, g_1) = f(g_0 g_1) - f(g_0) - f(g_1) = 0.$$

and if f is a bdd function,

$$[\overline{\delta}^1 f] = [0] \quad \text{in } H^2_b(\Gamma).$$

F is clearly a group homomorphism.

Also observe that $\mathrm{Im}(F) \subseteq \mathrm{Ker}(c^2 : H_b^2 \to H^2)$.

since $[\bar{\delta}^1 f] = 0$ in H^2 $\left(\begin{matrix}\text{it is non-trivial in } H_b^2 \\ \text{if } f \text{ is unbdd.}\end{matrix}\right)$.

<u>F is surjective onto $\mathrm{Ker}(c^2)$</u>:

If $c^2([\phi]) = 0$, then,

$$\phi = \bar{\delta}^1 h \quad \text{for some function } h : \Gamma \to \mathbb{R}$$

But $[\phi] \in H_b^2$, that is, $\phi : \Gamma^2 \to \mathbb{R}$ is bdd.

Hence, $\sup\limits_{\substack{g_0, g_1 \\ \in \Gamma}} |\phi(g_0, g_1)| < \infty$

$$\Rightarrow \sup_{\substack{g_0, g_1 \\ \in \Gamma}} \left| h(g_0 g_1) - h(g_0) - h(g_1) \right| < \infty$$

$$\Rightarrow h \in \widetilde{QM}(\Gamma) \text{ and } F([h]) = [\phi].$$

$$\widetilde{QM}(\Gamma) / \mathrm{Ker}\, F \cong \mathrm{Ker}\left(H_b^2(\Gamma) \xrightarrow{c^2} H^2(\Gamma)\right).$$

But $[f] \in \mathrm{Ker}\, F$

$\Rightarrow [\bar{\delta}^1 f] = 0$

$\Rightarrow \bar{\delta}^1 f = \bar{\delta}^1 g$ for some $g : \Gamma \to \mathbb{R}$, g bdd

$\Rightarrow \bar{\delta}^1 (f - g) = 0$

$\Rightarrow f - g = g'$, where $g' : \Gamma \to \mathbb{R}$ is a homomorphism is a ... ,

$$\Rightarrow \quad f = g' + g$$

$$\Rightarrow \quad [f] = [0] \quad \text{in } \widetilde{QM}(\Gamma).$$

III Cohomological characterization of higher rank symmetric spaces

Thm (Burger-Monod) : Let Γ be an irred. lattice in a higher rank semi-simple Lie gp G (with finite center). Then $\widetilde{QM}(\Gamma) = \{0\}$.

Converse (under some restrictions).

Thm (Bestvina - Fujiwara) : M = complete R-mfld, non-positive curvature, finite volume.

Assume $\Gamma = \pi_1(M)$ finitely generated, not virtually $\mathbb{Z}$, not virtually Cartesian product of infinite groups.

Then, $\widetilde{QM}(\Gamma) = 0 \Leftrightarrow M = \frac{\tilde{M}}{\Gamma}$ and $\tilde{M}$ is a higher rank symmetric space.

Proof uses Rank Rigidity Thm + a thm due Bestvina - Fujiwara.

Main Thm (B-F) : Let $X = CAT(0)$. $\Gamma \leq Iso(X)$, Γ acts properly discontinuously on X. Assume Γ is not virtually $\mathbb{Z}$, and Γ contains a rank 1 isometry.

Then, $\widetilde{QM}(\Gamma)$ = infinite dimensional.

Proof of Thm using the Main Thm :

Consider the de Rham decomposition of $\widetilde{M}$.

If $\widetilde{M}$ has no Euclidean factors, Rank Rigidity thm tells us one of the following must hold

(1) $\widetilde{M}$ is a higher rank symmetric space

(2) A deck transformation acts on $\widetilde{M}$ as rk 1 isometry.

(3) M has a finite cover that splits as $M' \times M''$. (Riemannian product)

Observe that (3) is not possible

$\pi_1(M)$ virtually $\pi_1(M') \times \pi_1(M'')$

where both M', M'' are non-positive curvature Rie. mflds. So, both $M, M'' \underset{\text{(diffeo)}}{\cong} \mathbb{R}^k$ for some k (not nec. same k.)

So, $\pi_1(M'), \pi_1(M'')$ cannot be finite.

If $\pi_1(M') = 0$, then, $M' \underset{\text{diffeo}}{\cong} \mathbb{R}^k \Rightarrow$ (Want to say "simply conn, non-pos curved $\Rightarrow$ infin vol, hence contradiction)

Hence, $\pi_1(M'), \pi_1(M'')$ are both infinite

$\Rightarrow \Gamma$ product of infinite grps (contradiction).

So, either (1) or (2) is true.

If (1) is true, $\widetilde{QM}(\Gamma) = \{0\}$ by Burger-Monod

If (2) is true, $\widetilde{QM}(\Gamma)$ infinite dim by Main Thm.

Hence, theorem has been proved.

Now, we show that given the conditions on $\Gamma = \pi_1(M)$, M cannot have Euclidean factors.

If $\tilde{M}$ is completely Euclidean, Γ is a Bierberbach gp. $\Rightarrow$ Γ contains some $\mathbb{Z}^n$ as finite index subgroup., contradiction.

$\tilde{M}$ could otherwise be $\underset{\substack{\uparrow \\ \text{Euclidean} \\ \text{factor}}}{\tilde{N}_0} \times \underset{\substack{\uparrow \\ \text{Non-Euclidean.}}}{\tilde{N}_1}$

Then, $\exists$ finite cover M' of M such that

$M' = T \times N$ where T = torus, N = non-positively curved mfld of finite volume.

Hence, $\pi_1(M)$ virtually contains some $\mathbb{Z}^k$, a contradiction.

Another application:

An application of Main Thm

$\Rightarrow$ If $G \subseteq MCG(S)$ is not virtually abelian, then $\widetilde{QM}(G)$ = infinite dimensional.

Appl: If Γ is an irred higher rk lattice,

then $\rho : \Gamma \longrightarrow MCG(S)$ a representation

$\Rightarrow \rho$ has finite image.