Volume Entropy

Abstract

I will introduce the notion of volume entropy of a Riemannian manifold and establish a relationship with the topological entropy of the manifold. I plan to sketch Manning's proof of the equality of these two entropies under the assumption of non-positive sectional curvature. To illustrate the importance of this concept of volume entropy, I will try to discuss a theorem due to Besson, Courtois and Gallot. Among other things, it will show that on a manifold, the locally symmetric metric of negative curvature is uniquely determined by two numbers - its volume entropy and volume.

Preliminaries

Geodesic Avoiding technicalities, can think of them as locally length minimizing curves.

Geodesic Flow It is a one parameter group $\{\phi_t\}$ of maps on TM given by the following – For a point (p, v) in TM, $\phi_t(p, v) = (\gamma(t), \dot{\gamma}(t))$ where $\gamma(t)$ is the geodesic through point p with velocity v. In this talk, we will concentrate on the geodesic flow on the unit tangent bundle and will denote it by SM.

For a Riemannian manifold (M, g), can define distance between any two points x, y by $d(x, y) = \inf l(\gamma)$ where the infimum is taken over all piecewise differentiable curves γ that join x and y. With this distance function, (M, d) becomes a metric space and this metric topology coincides with the original topology of M.

Complete manifold The geodesics are defined for all $t \in \mathbb{R}$. Once we have this, geodesic flow is defined for all $t \in \mathbb{R}$.

Result If M is a complete manifold, its universal cover $\tilde{M}$ is also complete. It is a Riemannian manifold equipped with the pullback metric from M.

Definitions

Topological Entropy (of the geodesic flow) - denoted by $h_{top}(\phi)$

Two equivalent definitions :

Definition 1 $W \subset SM$ is a (T, δ) separated set for geodesic flow if for any $x, y \in W$, there exists some t, $0 \leq t \leq T$, such that $d(\phi_t(x), \phi_t(y)) \geq \delta$. Let $N(T, \delta)$ be the cardinality of a maximal (T, δ) separated set. Then

$$h_{top}(\phi) = \sup_{\delta>0} \lim \sup_{T\to\infty} \frac{1}{T} \log(N(T, \delta))$$

Definition 2 $Z \subset SM$ is a (T, δ) spanning set for the geodesic flow if for each $x \in SM$, there exists $z \in Z$ such that $d(\phi_t(z), \phi_t(x)) \leq \delta$ for all $0 \leq t \leq T$. Then,

$$h_{top}(\phi) = \sup_{\delta > 0} \lim \sup_{T \to \infty} \frac{1}{T} \log(M(T, \delta)).$$

Note When we talk about (T, δ) separation in SM, we can use any metric on SM because the topological entropy is independent of the metric used.

Volume Entropy - denoted by $h_{vol}(g)$

Let $\tilde{M}$ be the universal cover of the manifold M. Volume entropy measures exponential growth rate of volume of closed balls in the universal cover $\tilde{M}$.

$$h_{vol}(g) = \lim_{R \to \infty} \frac{1}{R} \log(Vol(B(x, R)))$$

Manning's Results

Theorem 1

For a compact manifold M, the limit in the definition of volume entropy exists, is a non-negative quantity and is independent of the choice of the point x.

Proof sketch

Observe that

$$B(x, r + s) \subset \cup_{y \in B(x,r)} B(y, s).$$

Fix any b and look at the maximal subset of Y_r of $B(x, r)$ such that any two points in Y_r are atleast b distance apart. Hence

$$B(x, r + s) \subset \cup_{y \in Y_r} B(y, s + b).$$

Taking volumes and doing some computation,

$$\log Vol(B(x, r)) \leq \log Vol(B(x, s)) + k \log Vol(B(x, s + A))$$

where $k \leq \frac{r}{s} < k + 1$ and A is some constant. Taking $\lim \sup$ and $\lim \inf$ of this inequality proves the existence of the limit.

Independence from the choice of x is a consequence of tiranlge inequality. Non-negativity is obvious.

So, from now on, our manifold M will be compact. Thus, the universal cover $\tilde{M}$ will be a complete manifold.

Theorem 2

For a compact manifold (M, g), $h_{top}(\phi) \geq h_{vol}(g)$.

Proof sketch

Step 1 Take any r, δ and look at the annular region $B(x, r + \frac{\delta}{2}) \setminus B(x, r)$ in the universal cover. Let Q_r be a maximal subset with the property that any two points are 2δ apart. For each $q \in Q_r$, take the geodesic that connects x to q and let v_q be its initial velocity. The collection (x, v_q) for all $q \in Q_r$ forms a (r, δ) separated set for the geodesic flow on $\tilde{M}$.

Reason : Take (x, v) and (x, u). Can find q_1, q_2 in Q_r with $d(\pi\phi_r(x, v), q_1) < \delta/2$, $d(\pi\phi_r(x, u)) < \delta/2$. Clearly, $d(q_1, q_2) \geq 2\delta$. Then

$$d_1(\phi_r(x, v), \phi_r(x, u)) \geq d(\pi\phi_r(x, v), \pi\phi_r(x, u)) > \delta,$$

(d_1 is any metric on SM).

Step 2 Look at $(\pi(x), d\pi(v_q))$ for all those points (x, v_q) as above. This will form a (r, δ) separated set for geodesic flow in SM as geodesics that are δ apart in $\tilde{M}$ are at least δ apart in M. Thus,

$$h_{top}(\phi) \geq \limsup_{r\to\infty} \frac{1}{r} \log(\#Q_r)$$

Step 3 Can find a sequence $r_n \rightarrow infty$ such that $Vol(B(x, r_n + \frac{\delta}{2}) - Vol(B(x, r)) \geq \exp(\lambda - \varepsilon)r_n$ where λ is the volume entropy. We will take the limit in previous step along this subsequence. Also, for such an r_n,

$$(\#Q_{r_n})c_\delta \geq Vol(B(x, r_n + \frac{\delta}{2}) - Vol(B(x, r_n) \geq \lambda - \varepsilon.$$

This proves

$$h_{top}(\phi) \geq \lambda.$$

Application of Theorem 2

ASIDE : For a compact manifold, if all sectional curvatures are positive, then we have a uniform positive lower bound on Ricci curvature. Then, by Bonnet-Myers theorem, the universal cover $\tilde{M}$ is also compact and $\pi_1(M)$ is finite.

More generally, all compact manifolds have finitely generated fundamental groups. Let Γ be a finite generating set of $\pi_1(M)$. Let $w(k)$ denote the number of elements of $\pi_1(M)$ that can be expressed as a word in Γ using k or fewer characters. Define $\mu = \lim_{k\to\infty} \frac{1}{k} \log(w(k))$. Milnor had proved that this limit always exists. He had also proved that if all sectional curvatures are non-negative, then $w(k)$ has at most polynomial growth and if all sectional curvatures are negative, then $w(k)$ has exponential growth.

Let N be the Dirichlet fundamental domain. It determines a set of generators $\Gamma = \{g \in \pi_1(M) : gN \cap N \neq \emptyset\}$.

Application For a compact manifold M, $h_{top}(\phi) \geq \frac{\mu}{a}$ where μ is growth rate of $\pi_1(M)$ and $a = diam(N)$.

Proof Consider $B(x, r)$ where $x \in N$. Then gN is contained in $B(x, r)$ whenever the word length of g is $\leq \frac{r}{a} - 1$. Thus, $Vol(B(x, r) \geq Vol(N)w(\frac{r}{a} - 1)$ (slight abuse of notation here) which proves the result.

Corollary If a manifold admits a metric with all sectional curvatures negative, topological entropy of the geodesic flow would be positive.

Proof If all sectional curvatures negative, by Milnor's result, μ is greater than 1. Scale the metric in such a way that $a = 1$. μ is independent of the metric. So, $h_{top}(\phi) > 0$.

Theorem 3

If all sectional curvatures of M are non-positive, then $h_{top}(\phi) = h_{vol}(g)$.

Proof sketch

We will use the following metric d_1 on SM : For (p, v) and (q, u) in SM, let σ, $\tau : [0, 1] \to M$ be the geodesics through p, q with initial velocities v, u respectively. Then, the d_1 distance between them will be given by

$$\sup_{t \in [0,1]} d(\sigma(t), \tau(t)).$$

Step 1 **(The role of negative sectional curvature)** In negative sectional curvature, the geodesics starting at same point move away exponentially fast and the supremum of the distance between them occurs at the endpoint. So, the d_1 distance (as defined above) is bounded above by $d(\sigma(0), \tau(0)) + d(\sigma(R), \tau(R))$ over the interval $[0, R]$.

Step 2 Let N be a compact fundamental domain in $\tilde{M}$ of diameter a. Fix any r, δ. Let $A_r = \{z \in \tilde{M} : r - a \leq d(z, N) \leq r\}$ and C_r be a maximal subset of A_r with the property that any two points are at least δ apart. Then

$$(\#C_r)c_\delta \leq \exp((\lambda + \varepsilon)(r + a + \delta)).$$

Let E be a maximal subset of N with the property that any two points are at least δ apart.

Step 3 Take the unique geodesics that connect points of E with points in C_r (geodesics in $\tilde{M}$). If σ is such a geodesic, look at the point $(\sigma(0), \dot{\sigma}(0))$ and project it down to SM. We will show that this collection in SM is a $(r - 1, 4\delta)$ spanning set for the geodesic flow on SM.

Step 4 Take any geodesic of length r, between u_0, v_0 in M. Lift it to a geodesic in $\tilde{M}$ between u and v where u is in N. By maximality in choice of E and C_r, we can choose $y \in E$ and $z \in C_r$ such that $d(y, u) \leq \delta, d(z, v) \leq \delta$. Then, the geodesic between y and z has length : $r - 2\delta \leq d(y, z) \leq r + 2\delta$. Select the point z' in this geodesic such that $d(y, z') = r$. Then, show that d_1 distance between initial velocity vectors of uv and yz' is bounded above by

$d(u,y)+d(v,z')\leq 4\delta$. Project these geodesics down to M and suppose y_0z_0 is the projection of yz. Then, the d_1 distance in SM between initial velocity of u_0v_0 and y_0z_0 is $\leq 4\delta$.

So, the pushforward of the geodesics yz from $\tilde{M}$ produces a $(r-1, 4\delta)$ spanning set for the geodesic flow on SM and the number of such geodesic in $\tilde{M}$ is $(\#E)(\#C_r)$. Finally,

$$h_{top}(\phi) \leq \limsup_{r\to\infty} \frac{1}{r}\log(\#C_r\#E) \leq \lambda+\varepsilon.$$

A look at volume entropy

The volume entropy is not a very good geometric quantity as it is not invariant under rescaling of Riemannian metrics

$$h_{vol}(\lambda g) = \frac{1}{\sqrt{\lambda}} h_{vol}(g).$$

However, the quantity $h^n_{vol}(g)Vol(M,g)$ is a scale invariant dynamical quantity. Some authors call it the **normalized entropy**.

Katok had proved results connecting normalized entropy with metric entropy for geodesic flows. But I will not about that.

Gromov had proved that for any metric g on M,

$$h^n_{vol}(g)Vol(M,g) \geq \frac{1}{C_n n!}||M||$$

where $||M||$ is the simplicial volume of M (computed only from topological data of M) and $C_n = \dfrac{\Gamma(n/2)}{\sqrt{\pi}\Gamma((n+1)/2)}$.

Now if M carries a metric of constant negative curvature, say g_0, then $||M||$ can be computed explicitly in terms of $Vol(M,g_0)$. In that case, we can find a constant C'_n such that for any g,

$$h^n_{vol}(g)Vol(M,g) \geq C'_n h^n(g_0)Vol(M,g_0).$$

Based on these, Gromov had conjectured that the if M has a locally symmetric metrics of negative curvature (say g_0), then the normalized entropy of (M,g_0) is the minimum possible normalized entropy.

Besson-Courtois-Gallot proved this conjecture by proving the following celebrated theorem :

Theorem 4 [BCG] Let (N,g) and (M,g_0) be two compact manifolds of dimension n where g_0 is locally symmetric with negative curvature and $f : N \to M$ is a continuous map with $\deg(f)\neq 0$. Then,

$$h^n_{vol}(g)Vol(N,g) \geq |\deg(f)| h^n_{vol}(g_0)Vol(M,g_0).$$

If $n \geq 3$, then the equality is achieved only when (N, g) is locally symmetric and there exists a positive λ such that $(N, \lambda g)$ is a Riemannian covering of (M, g_0) with the covering map homotopic to f.

NOTE By homothety (rescaling by positive constant λ), the equality is equivalent to having $Vol(N, \lambda g) = |\deg(f)| Vol(M, g_0)$ and $h^n_{vol}(\lambda g) = h^n_{vol}(g_0)$.

Application Replace N in the above theorem by M itself and f by id. Let g be any metric on M and g_0 be a locally symmetric metric of negative curvature. Then $Vol(M, g) = Vol(M, g_0)$ and $h^n_{vol}(g) = h^n_{vol}(g_0)$ if and only if (M, g) is a Riemannian covering of (M, g_0) that is homotopic to id. Thus, (M, g) is isometric to (M, g_0).

This shows that the locally symmetric metrics of negative curvature are uniquely determined by the volume and volume entropy.

Major Application of BCG In the proof of Mostow Rigidity theorem.

Statement of Mostow Rigidity theorem

- (Geometric statement) Let N and M be compact manifolds of dimension $n \geq 3$ and constant negative sectional curvature with $f_* : \pi_1(M) \to \pi_1(N)$ isomorphism of groups. Then, N and M are isometric (upto rescaling of metrics) and f is homotopic to this isometry.

- (Algebraic statement) Let Γ and Δ be lattices in $SO(n, 1)$ where $n \geq 3$ and $f : \Gamma \to \Delta$ is a group isomorphism. Then, there exists $g \in SO(n, 1)$ such that $\Delta = g\Gamma g^{-1}$.

Proof of Geometric version

As (N, g) and (M, g_0) are aspherical manifolds (using Cartan-Hadamard theorem), f_* is an isomorphism of all homotopy groups. Thus, by Whitehead's theorem (Weak homotopy equivalence implies homotopy equivalence), f is a homotopy equivalence. Thus, $|\deg(f)| = 1$. As both g and g_0 are hyperbolic metrics in this case,

$$h^n_{vol}(g) Vol(N, g) = h^n_{vol}(g_0) Vol(M, g_0).$$

Let's rescale g such that $Vol(N, \lambda g) = Vol(M, g_0)$. Then, by BCG, $(N, \lambda g)$ is isometric with (M, g_0).