PROSEMINAR/SEMINAR: MATRIX (LIE) GROUPS

Lecturer: Dr. Mitul Islam
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Meeting Time: Thursdays 16.00 - 18.00
Meeting Place: Seminarraum 8, Mathematikon
Language: The talks are supposed to be in English.

Abstract

This is a seminar targeted at mid-level as well as advanced undergraduate students. The goal of the course is to introduce some basic ideas in Lie theory through matrix groups. There will be a strong emphasis on looking at plenty of examples of matrix groups.

The first part of the course will be a fairly general introduction to matrix groups and their Lie algebras - both as algebraic objects and as tangent spaces at identity to the respective groups. In the second part, we will focus on compact matrix groups and discuss their structure theory. At the end, we will cover the classification theorem of compact semi-simple Lie groups and look at some concrete examples.

Recommended Pre-requisite

Linear Algebra, Analysis (Basic topology in $\mathbb{R}^n$ and the notion of derivatives), Basic abstract algebra (group theory).

Roughly speaking, if you feel comfortable with the first three chapters of Reference (1) below, this class is well-suited for you.

Key References

(1) **(A very simple introductory textbook)** Matrix Groups for undergraduates, Kristopher Tapp (AMS). (**only for talks 1-6**)
(2) **(Main Reference Text)** Matrix Groups: an Introduction to Lie Group Theory, Andrew Baker, Springer, 2002.

Rules

Grading. You must give a talk to receive a grade. The quality of your talk will determine your grade. Please register on Müsli for the class, if you haven't.

If for some reason, you are unable to give the talk that you had signed up for, you should let me know at least two weeks ahead of your scheduled talk. Otherwise, you risk attracting a grade penalty. This of course does not apply to an emergency situation (e.g. medical or family). In case of such emergencies, a decision will be made on a case-by-case basis.

Participation. While attendance and participation isn't mandatory, you will receive bonus grade points for active participation in the seminar.

Date: February 2023.

Seminar Presentation

Preparing for the talk. Ideally one should start preparing for their talk around 3-4 weeks ahead of time. Suppose your talk is on the day T.

- Between $T-20$ to $T-8$, you should have a meeting with me to discuss the material of your talk: give me a short overview of your talk, ask me questions about the material, etc.
- Between $T-7$ and $T-3$, you should either send me your talk notes or have a second meeting to give me a 10-minute version of your talk.

Day of the talk. Presentations using a board is preferred. But if you have other ideas like using a tablet or slides, please discuss this with me ahead of time.

For your talk, you are expected to prepare two things besides your presentation:

- a 1-page handout (could be typed, written on a tablet, legibly hand-written, etc.) that contains a short list of things you are covering. You can send me this before the talk and I can make copies for everyone.
- a problem hand-out with 1 or 2 problems based on your presentation.

The talk. You should plan to speak for 35-40 minutes. Then you ask the class to work on one problem from your problem hand-out and give them a few minutes to brainstorm. Then there is a 5-minute session when there is some discussion and then you explain the solution.

Class time. Per the usual convention, we will start 15 minutes after the hour (i.e. 16.15 if the official start time is 16.00) and we will try to end 15 minutes before the two-hour mark (i.e. 17.45 if the official end time is 18.00). But if the last talk is going over the expected time, please be patient and do not interrupt the speaker.

There will often be two talks on the same day. On those days, there will be a 5-minute break between the two talks.

Questions. You are strongly encouraged to ask questions - both at the end during discussions as well as during the talk. You must not deter yourself from asking questions because you are worried that you will interrupt the talk.

Schedule

Date	Talk
April 20	Talks 1 and 2
April 27	Talk 3
May 4	Talk 6
May 11	Talks 4 and 7
May 18	–
May 25	Talks 9 and 5
June 1	–
June 8	–
June 15	Talk 8
June 22	–
June 29	Talks 10 and 11

LIST OF TALKS

1. Introduction to Matrix Groups and Examples

Emilian Arnold

We will introduce the following families of matrix groups which will serve as the main examples throughout the course.

Key examples: $\mathrm{GL}_n(\mathbb{K})$, $\mathrm{SL}_n(\mathbb{K})$ for $\mathbb{K} = \mathbb{R}, \mathbb{C}$; orthogonal groups $O(n), U(n), SO(n), SU(n)$.

Explain how to think of complex matrix groups as real matrix groups.

Have a discussion on the group of isometries of the Euclidean space $\mathbb{R}^n$. Explain how orthogonal groups arise as groups that preserve certain bilinear forms.

It is also important to show some low-dimensional examples. Write down $SO(2)$ and $SO(3)$ and explain that it encodes positions on a cricle and a 2-dimensional sphere. Then write down $SU(2)$. Prove (or at least give some ideas about) the following: $SU(2)$ is a double cover of $SO(3)$.

Very important: give examples to show that not all groups are matrix groups (Section 7.7 of Baker's book (Ref 2) gives such an example)!

If time permits, explain the group of quarternions and the matrix groups over them. Explain how they can be realized as matrix groups over $\mathbb{C}$ and hence $\mathbb{R}$.

2. Topology of Matrix Groups

Friedrich Homann

Discuss the topology on matrix groups - open sets, closed sets, limit points, homeomorphisms, connectedness, compactness.

The ideal way to structure this class is to first say that we will think of matrix groups as subsets of $\mathbb{R}^m$ by looking at the coordinate map. Then introduce all these concepts in $\mathbb{R}^m$. Then explain how to give a matrix group the subspace topology.

It is important that you give examples of open matrix groups, closed matrix groups, compact matrix groups, connected and disconnected matrix groups.

3. Lie Algebra

Robin Campbell and Dominik Svorad

In this class, we will define the Lie algebra of a matrix group at the tangent space of the group at identity. You can structure your talk loosely following Chapter 5 of Tapp. The talks should encompass the following:

(a) Introduce Lie algebras as tangent space at identity. To do this, first introduce the necessary background - define tangent vectors and tangent space for subsets of the Euclidean spaces and then use this to define Lie algebras.
(b) You do NOT need to go into Lie brackets. That will be done in the next class. You can just briefly mention that Lie algebras also has a different formulation in terms of Lie brackets which will be done in the next lecture. In this talk, you will focus mostly on the tangent space viewpoint and do examples.
(c) Talk about the dimension of Lie groups using Lie algebras (Definition 3.17 of Baker)
(d) Compute examples (Section 3.3 of Baker or Chapter 5 of Tapp) of Lie algebras. Do some dimension computations as well.
(e) Explain how the Lie algebras can be thought of as the space of left-invariant vector fields (Chapter 5 Section 3 in Tapp).
(f) Talk a little bit about the complexification of Lie algebras, see Section 3.6 of Baker's book. In particular, discuss the example of Page 94-95. It is an example where two non-isomorphic real Lie algebras have the same complexification.

Note that examples are crucial. So please compute the Lie algebras in all of our key examples.

4. Matrix Exponentiation

Amelia Faber

This talk is going to be on the exponential map between Lie algebras and matrix groups. The talk could be structured loosely based on Chapter 6 of Tapp, supplemented by details from Chapter 2 of Baker. It should encompass the following:

(a) Define the exponential map exp for matrices. For this, you will first need to explain what it means for a series in a matrix group to converge. (Chapter 6 Section 2 of Tapp)
(b) Discuss some algebraic properties of this map, refer to Chapter 6 Section 4 of Tapp.
One important thing to mention here is that:
- $\exp(A + B) = \exp(A)\exp(B)$ only under some conditions, unlike the real-valued exponential map. For reference, see Proposition 2.2 of Baker's book (ref 2)
- Mention the Baker-Capmbell-Hausdorff formula (see e.g. Wikipedia article on this) that in general gives a formula for $\exp(A)\exp(B)$.
(c) Discuss that the map $\exp : M_n(\mathbb{K}) \to \mathrm{GL}_n(\mathbb{K})$, where $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, is a local diffeomorphism near 0. A reference for this could be Proposition 2.4 of Baker.
(d) Explain that because of the above, exp is a map that takes you from the Lie algebra to the matrix groups. (Chapter 6 Section 3 of Tapp).
(e) Further, the map exp has a local inverse: discuss the log function. Refer to Proposition 2.3 of Baker.
(f) Then discuss the connection of exp with matrix-valued ODEs. Interpret $x(t) := \exp(tA)$ as a solution of the ODE $x'(t) = A \cdot x(t)$ (like in the case of real variables). References are Sections 2.3-2.4 of Baker and Proposition 6.9-6.10 of Tapp.
(g) Finally discuss an application of the exp map to matrix-valued differential equations. Show an explicit computational example, e.g. pick one of the Examples 2.20-2.23 from Baker. Along the way, you might need to use the Jordan canonical form of matrices and why it's easy to compute the exponentials of Jordan matrices.

5. Matrix groups as manifolds

Simon Weiß

This talk is supposed to introduce the basic definitions of smooth manifolds with a focus on explaining the statement: "matrix groups are smooth manifolds where the group operations are smooth maps". The talk could be structured loosely based on Chapter 7 of Baker's book.

(a) Define the notion of a smooth manifold (using charts, atlases, and transition maps). Then explain the notion of tangent vectors (as derivatives of curves) and tangent space at a point. Give some good and concrete examples here - say sphere $\mathbb{S}^2$, torus, and cylinder.
(b) Then explain the notion of derivative for a map between two manifolds (before doing this, recall the notion of derivative for functions between $\mathbb{R}^m$ and $\mathbb{R}^n$). Give an example of the derivative of a function, say $f : \mathbb{S}^2 \to \mathbb{R}$ given by $f(x, y, z) = z$.
Then explain what a smooth map is.
(c) Explain that matrix groups have the structure of smooth manifolds. Remind the audience of the earlier definition of Lie algebras that they are tangent spaces at identity.
(d) Use this abstract notion of manifolds, to define Lie groups. Conclude the discussion by explaining that:
    (a) all matrix groups are Lie groups (so we have already seen many examples of Lie groups!)
    (b) but not conversely (remind the audience of the example from Talk 1 of a group that isn't a matrix group. It should be clear now that the example was a Lie group. The example came from Section 7.7 of Baker's book.)

6. Lie Bracket

Jonas Biba and David Barth

The goal of the talk is to define Lie algebras algebraically - as a vector subspace with a bilinear map called the Lie bracket. Then one should reconcile this with the definition we saw earlier as tangent space at identity. The talk could be loosely based on Chapter 8 of Tapp's book. The talk must encompass the following:

(a) Introduce the Lie bracket $[A, B]$ as the commutator and show that it satisfies bilinearity, alternativity, and Jacobi identity (see e.g. Wikipedia article on Lie algebras for help with these last three terms)
(b) Define Lie algebras as vector subspaces that carry a Lie bracket. In particular, explain why all the Lie algebras we had seen earlier (defined in the sense of tangent spaces) are Lie algebras in this sense.
(c) Introduce Lie algebra homomorphisms. Discuss a few things, like how the derivative of Lie group homomorphisms (define them now) produce Lie algebra homomorphisms. Discuss what happens

in the case of isomorphisms. An example to discuss here is $SO(3)$ and $SU(2)$. Their real Lie algebras are isomorphic although the groups are not (remind the audience about this example that we saw earlier; for reference see Section 3.5 of Baker or Pg 117 of Tapp).

(d) State the Lie correspondence theorem (Theorem 8.16 in Tapp's book, page 125). You can have an informal discussion about how one might prove such a thing (follow the discussion in Tapp). A precise proof would not be possible since we do not have "exponential" maps yet.

(e) Discuss the adjoint action (Chapter 8 Section 2 of Tapp) and explain the map $Ad : G \to \mathrm{GL}_n(\mathbb{K})$ that we will get for any matrix group G. This is the so-called adjoint representation.

Discuss the maps Ad and ad and their relationship (Proposition 8.10 of Tapp). Include an example of the adjoint action on a compact group, see Chapter 8 Section 3 on Tapp.

(f) (Time permitting) Remind the audience about the complexification of Lie algebras, following Section 3.6 of Baker. Then explain the content of Page 94 in Section 3.6 of Baker. Here you see two non-isomorphic real Lie algebras. But their complexified Lie algebra is the same (hence isomorphic).

Time permitting, define Lie algebras abstractly using an abstract operation $[,]$ satisfying some properties. And mention that these arise as tangent spaces to Lie groups (possibly, not a matrix group).

7. Lie groups and Homogeneous Spaces

Anton Tapking

You could structure this talk loosely based on Chapter 8 of Baker's book. Introduce the notion of matrix group actions on spaces and define homogeneous spaces.

(a) Introduce quotients G/H of matrix groups and explain the quotient topology on G/H. Refer to Section 8.1 of Baker.
(b) Realize homogeneous spaces as orbits spaces of group actions, see Section 8.2 of Baker.
(c) Explain what it means for a matrix group to act smoothly on a space (or a smooth manifold) and define homogeneous spaces. The point here is that homogeneous spaces are all group quotients as above.
(d) Do this example. Define real and complex projective spaces, $\mathbb{K}P^n$ or $\mathbb{P}(\mathbb{K}^{n+1})$ where $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. Realize them as homogeneous spaces.
(e) Compute the tangent space in the above example. Show that for a quotient $X = G/H$, the tangent space T_eX is a linear subspaces of the Lie algebra of G but it isn't usually a Lie algebra.
(f) (Time permitting) Discuss the Grassmanians as another example; see Section 8.4 of Baker.

8. Adjoint Representation

Philip Nazari and Alexander Marwitz

The goal of this talk is to discuss the adjoint representation of a compact Lie group and understand the structure of this representation. You can base your talk on Section 11 of Baker. The talk should encompass the following:

(a) Define the map Ad and ad. Some of this will have been introduced already but its good to recall them, albeit quickly.
(b) Define the Ad-invariant inner product $(X, Y) = tr(XY^*)$, Section 11.1 of Baker.
(c) Show that: if G is a subgroup of $O(n)$ or $U(n)$, then $Ad(G)$ is a subgroup of $SO(n)$ (using the above inner product). This is Theorem 11.2 of Baker. Also, explain Theorem 11.8 which gives more details about the image and kernel of Ad. *At this point, it is perhaps good to comment that the above point essentially covers all compact matrix groups as any compact matrix group can always be realized as a subgroup of $O(n)$ or $U(n)$. This is Theorem 10.1 of Baker.*
(d) It is important to do some examples here. Compute the adjoint representation of $SU(2)$. This produces the map $Ad : SU(2) \to SO(3)$, that we have often alluded to in this seminar (remind the audience). The computation of the $kerAd$ shows that this map is 2-to-1 (a double cover) while ad is a Lie algebra isomorphism. Section 3.5 (pg 86) of Baker does this specific computation. But you will not need many of these details since you have worked out the details in a lot more generality already.
(e) Now it is time to discuss the structure of the adjoint representation. We will discuss Section 11.5 of Baker.
- Suppose that G is a subgroup of $O(n)$ or $U(n)$.
- Recall that maximal tori have been defined in the previous talk. If g_0 is a topological generator, Ad_{g_0} action leads to a decomposition of $\mathfrak{g}$. Define what a root is and show that this above thing is

the root space decomposition. This is basically pg 278-280 of Baker. The main result is Theorem 11.21. You could then just mention Theorem 11.22 as what you get when you complexify.

(f) The above is a lot and your audience might be tired. So its time to do two concrete examples. Do two examples, say $SO(3)$ and $SU(2)$ with details, see Examples 11.26-29 of Baker. By 'details', I mean write down a maximal tori, the roots, the root spaces, the root space decomposition.

(g) You could also briefly mention the example of $SU(3)$, because there are many roots. Note that the situation looks quite different from above cases. This can be attributed to the rank of the group being ≥ 2, a concept that has been introduced in the previous talk.

(h) (Time permitting) Hopefully the examples have cleared up the key idea here. Now remind the audience that you have a Weyl group for these groups. Then, we can define a Weyl group action on the root space and also the dual root space, see pg 281 and Prop 11.23 of Baker. After mentioning this result, show what this means in your examples - what is this action in those cases.

9. Maximal Tori

Julia Piazolo

This talk will introduce the notion of maximal tori in a compact, connected matrix group. Recall for the audience the examples of compact Lie groups that we have been focusing on in this course - $O(n), SO(n), U(n), SU(n)$. The talk can be structured following Chapter 10 of Baker (ref 2). Chapter 9 of Tapp's book also could be useful for helping your understanding, but I will mostly put references to Baker in the following. Your talk should at least include the following:

(a) Introduce the notion of a standard r-torus $\mathbb{T}^r$. Note that $\mathbb{T}^r \cong \mathbb{R}^r / \mathbb{Z}^r$. (Section 10.1 of Baker)

(b) Define what is a torus in general (any group isomorphic to a standard torus). Note that every torus has a topological generator (Prop 10.7 of Baker).

(c) Define maximal torus in a matrix group, Definition 10.8 of Baker. Show some examples of maximal tori, e.g. Pg 255 and Proposition 10.10 of Baker.

(d) Explain the relationship between various maximal tori in compact connected matrix groups and how each element lies in a maximal tori (Theorems 10.11-10.12, 10.13 of Baker). The principal axis theorem 10.13 is a nice thing to cover here.

(e) By Lie correspondence theorem, these maximal tori correspond to some Lie algebras. Discuss the principal axis theorem for these Lie algebras (Theorem 10.15 of Baker).

(f) Define the notion of rank of a compact Lie group (see Chapter 9 Section 6 of Baker for this definition).

(g) Explain that the maximal tori are in fact maximal abelian subgroup, Theorem 10.19 of Baker.

(h) Define the notion of the Weyl group of a maximal torus (pg 260). Explain the properties of Weyl group - finite, acts faithfully on the maximal torus by conjugation. (Theorem 10.21)

(i) Do some concrete computations of maximal tori and the Weyl groups - like $SO(3), SU(2), U(2), SU(3)$. You can see Baker's Examples 11.26-11.29 in Chapter 11 pg 282-287 for these examples. You should do at least three of these examples.

(j) (Time permitting) Explain that maximal tori help you compute the center of matrix groups, see Section 10.4 of Baker.

10. Classification of Compact Matrix groups: Roots Systems and Dynkin Diagrams

Mitul Islam

Explain what is an abstract root system.

Then quickly move to examples. It is important that you do the examples $A_1, A_2, B_2, A_1 \times A_1$ at least.

Then introduce the notion of Dynkin diagrams and draw them for the above ones.

Then show them the classification table to show much more exists. But we are not going to go into further details.

Then finally say how to go from the data of a root system back to Lie algebra.

Do a few concrete examples - maybe A_1, A_2, and B_2.

In the end, show the classification table of compact simple Lie groups. And say a few words about how this gives a complete list.

11. Introduction to (Compact) Lie Groups

Mitul Islam

A glimpse at what lies beyond.